COMMUTING OPERATORS AS AN INSTANCE OF ITERATIVE, GENERALISED HASSE–SCHMIDT RINGS

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In [1], Rahim Moosa and Thomas Scanlon define $\mathcal{D}$-rings, or rings with free operators. Fixing a base field k, a finite-dimensional k-algebra $\mathcal{D}$, and a k-algebra homomorphism $\pi\colon \mathcal{D} \to k$, a $\mathcal{D}$-ring is a k-algebra R equipped with a k-algebra homomorphism $R \to \mathcal{D}(R) = R \otimes_k \mathcal{D}$ which is a section to $\mathrm{id}_R \otimes \pi$. This condition is equivalent to saying that a $\mathcal{D}$-ring structure on R is a finite sequence of k-linear operators ∂_i that satisfy some product rule that depends on $\mathcal{D}$. No other equations are satisfied by the ∂_i. In particular, they do not need to pairwise commute.

In section 6 of [3], the authors construct a generalised Hasse–Schmidt system $\underline{\mathcal{D}}$ with an iteration system Δ (as defined in [2]) such that the Δ-iterative $\underline{\mathcal{D}}$-rings are precisely the $\mathcal{D}$-rings. This allowed them to use the geometric methods of Hasse–Schmidt subschemes they developed in [2] in their analysis of finite-dimensional minimal types in the theory $\mathcal{D}$-CF_0.

In this short note, I will construct a generalised Hasse–Schmidt system $\underline{\mathcal{D}}$ with an iteration system Δ such that the Δ-iterative $\underline{\mathcal{D}}$-rings are precisely the $\mathcal{D}$-rings *where all the operators ∂_i pairwise commute*. The idea is similar to that of [3].

Fix a k-basis $\varepsilon_0, \ldots, \varepsilon_l$ of $\mathcal{D}$. Given the standard k-algebra structure $s\colon k \to \mathcal{D}$ and the residue map $\pi\colon \mathcal{D} \to k$, define the following:

- $\mathcal{D}^{(n+1)} = \mathcal{D} \circ \mathcal{D}^{(n)}$;
- $s_{n+1} = s^{\mathcal{D}^{(n+1)}} \circ s$;
- $\pi_{n+1} = \pi^{\mathcal{D}^{(n)}} \colon \mathcal{D}^{(n+1)} \to \mathcal{D}^{(n)}$.

If we identify $\mathcal{D} \circ \mathcal{D}^{(n)} = \mathcal{D}^{(n)} \otimes_k \mathcal{D}$, then $\pi_{n+1} = \mathrm{id}_{\mathcal{D}^{(n)}} \otimes \pi$.

We now set up some notation. Let $I = \{(i_1, \ldots, i_n)\colon 0 \leq i_j \leq l\}$. For each $\underline{i} \in I$, write $\varepsilon_{\underline{i}} = \varepsilon_{i_1} \otimes \ldots \otimes \varepsilon_{i_n} \in \mathcal{D}^{(n)}$. Then $\{\varepsilon_{\underline{i}}\colon \underline{i} \in I\}$ forms a basis of $\mathcal{D}^{(n)}$.

In [3], this $\mathcal{D}^{(n)}$ was too big; it did not take into account the fact that ∂_0 must be the identity. Here we have a similar issue: it does not take into account the fact that the ∂_i must pairwise commute (and have $\partial_0 = \mathrm{id}$). So we define the equivalence relation $\sim$ on I by $\underline{i} \sim \underline{j}$ if and only if they are the same tuple up to reordering. Then define

$$\mathcal{D}_n = \left\{ \sum_{\underline{i} \in I} r_{\underline{i}} \varepsilon_{\underline{i}} \in \mathcal{D}^{(n)} \colon r_{\underline{i}} = r_{\underline{j}} \iff \underline{i} \sim \underline{j} \right\}.$$

For each permutation $\sigma \in S_n$, define the following map:

$$\begin{aligned} \Gamma_\sigma \colon \mathcal{D}^{(n)} &\to \mathcal{D}^{(n)} \\ \varepsilon_{\underline{i}} &\mapsto \varepsilon_{\sigma \underline{i}} \end{aligned}$$

We now claim that $\mathcal{D}_n$ is the equaliser of the maps $(\Gamma_\sigma)_{\sigma \in S_n}$. Note that $\mathcal{D}_n$ is the *setwise* equaliser of the $(\Gamma_\sigma)_{\sigma \in S_n}$. Let $F\colon \mathsf{Alg}_k \to \mathsf{Set}$ be the forgetful functor from the category of k-algebras to the category of sets. Then F has a left adjoint: the free k-algebra construction. So F preserves limits, and thus $\mathcal{D}_n$ must be the equaliser in Alg_k.

We now check that $\pi_n\colon \mathcal{D}^{(n)} \to \mathcal{D}^{(n)}$ restricts to a surjective morphism $\pi_n\colon \mathcal{D}_n \to \mathcal{D}_n$, and that $\mathcal{D}_{m+n} \subseteq \mathcal{D}_m \circ \mathcal{D}_n$ as subalgebras of $\mathcal{D}^{(m+n)}$. Let $\Delta_{(m,n)}$ be these inclusion maps. Then it is immediate that $(\underline{\mathcal{D}}, \Delta)$ is an iterative Hasse–Schmidt system in the sense of Definition 2.17 of [2].

One then readily checks that an iterative $\underline{\mathcal{D}}$-ring corresponds precisely to a $\mathcal{D}$-ring structure where the operators pairwise commute.

References

[1] Rahim Moosa and Thomas Scanlon. Jet and prolongation spaces. *Journal of the Institute of Mathematics of Jussieu*, 9(2):391–430, 2010.

[2] Rahim Moosa and Thomas Scanlon. Generalized Hasse-Schmidt varieties and their jet spaces. *Proceedings of the London Mathematical Society*, 103(2):197–234, 2011.

[3] Rahim Moosa and Thomas Scanlon. Model theory of fields with free operators in characteristic zero. *Journal of Mathematical Logic*, 14(2):1450009, 2014.