Algebraic Number Theory 2019-20

Example Sheet 2

Hand in the answers to questions 3 and 5 (marked with †).
Deadline 12 noon Monday, Week 6 (4 November)
For questions about the example sheet, it is best to ask them on Moodle. Questions must be asked before 5 pm on Friday to get an answer before the deadline.

1. Let $f(X) = X^3 - 3X - 3$. Let α be a root of f and let $K = \mathbb{Q}(\alpha)$.
   (i) Show that f is irreducible.
   (ii) Let $\beta = 1 + \alpha + \alpha^2$. Work out the matrix of $m_{K,\beta}$ in the basis $\{1, \alpha, \alpha^2\}$.
   (iii) Calculate the characteristic polynomial $\chi_{K,\beta}(X)$ and deduce that the minimal polynomial $\mu_{\mathbb{Q},\alpha}(X)$ is $X^3 - 9X^2 + 12X - 7$.

2. Let $L = \mathbb{Q}(\sqrt[3]{d})$. Show that the basis $\{1, \sqrt[3]{d}, \sqrt[3]{d^2}\}$ for L has discriminant $-27d^2$. (Try doing this using both formulae for the discriminant.)

†3. Let $f(X) = X^3 - X^2 - 3$. Let α be a root of f and let $K = \mathbb{Q}(\alpha)$.
   (i) Show that f has no roots in $\mathbb{Z}$. Deduce that f is irreducible over $\mathbb{Q}$.
   (ii) Compute the matrix m_{K,α^2} with respect to the basis $\{1, \alpha, \alpha^2\}$.
   (iii) Write down the traces $\mathrm{Tr}_{K/\mathbb{Q}}$ of $1, \alpha, \alpha^2$.
   (iv) By using the equation $\alpha^3 = \alpha^2 + 3$ and the fact that $\mathrm{Tr}_{K/\mathbb{Q}}$ is $\mathbb{Q}$-linear, or otherwise, calculate $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^3)$. Show that $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha^4) = 13$.
   (v) Calculate $\Delta(1, \alpha, \alpha^2)$ and deduce that $\{1, \alpha, \alpha^2\}$ is an integral basis for K.
   (vi) Explain in two or three sentences why $K \neq \mathbb{Q}(\sqrt[3]{d})$ for any $d \in \mathbb{Q}$. (You may use the result of Q2, as well as any results from lectures.)

4. Suppose $f(X) = X^3 + bX + c \in \mathbb{Q}[X]$ is irreducible and let α be a root. Let $K = \mathbb{Q}(\alpha)$. Show that

$$\Delta(1, \alpha, \alpha^2) = -4b^3 - 27c^2.$$

†5. Let p be an odd prime and let $\zeta = \zeta_p = \exp(2\pi i/p)$. Let $f(X) = X^{p-1} + X^{p-2} + \cdots + X + 1$ be the minimal polynomial of ζ. Let $K = \mathbb{Q}(\zeta)$ and let $\omega = \zeta - 1$.
   (i) Use the minimal polynomial of ω to prove that $\mathrm{Nm}_{K/\mathbb{Q}}(\omega) = p$.
   (ii) Explain why the conjugates of ζ are

$$\zeta,\ \zeta^2,\ \zeta^3,\ \ldots,\ \zeta^{p-1}.$$

   (iii) Using the determinant of a Vandermonde matrix, show that

$$\Delta(1, \zeta, \ldots, \zeta^{p-2}) = \prod_{1 \leq i < j \leq p-1} (\zeta^i - \zeta^j)^2 = (-1)^{(p-1)/2} \cdot \prod_{\substack{1 \leq i,j \leq p-1, \\ i \neq j}} (\zeta^i - \zeta^j).$$

(iv) Prove that

$$\Delta(1, \zeta, \ldots, \zeta^{p-2}) = (-1)^{(p-1)/2} \left(\prod_{i=1}^{p-1} \zeta^i \right)^{p-2} \cdot \left(\prod_{k=1}^{p-1} (\zeta^k - 1) \right)^{p-2}.$$

Express this in terms of $\mathrm{Nm}_{K/\mathbb{Q}}(\zeta)$ and $\mathrm{Nm}_{K/\mathbb{Q}}(\omega)$ and deduce that

$$\Delta(1, \zeta, \ldots, \zeta^{p-2}) = (-1)^{(p-1)/2} p^{p-2}.$$

(v) Using the fact that the minimal polynomial of ω is Eisenstein at p, show that $\frac{\omega^{p-1}}{p}$ is an algebraic integer.

(vi) By calculating the norm or otherwise, prove that if $t \in \mathbb{Z}$ and $0 < t < p$ then $\frac{t\omega^{p-2}}{p}$ is not an algebraic integer.

6. Let $f(X) \in \mathbb{Q}[X]$ be a monic irreducible polynomial of degree n. Let the roots of f in $\mathbb{C}$ be $\alpha_1, \ldots, \alpha_n$. Write $\alpha = \alpha_1$ and $K = \mathbb{Q}(\alpha)$.

(i) Differentiate $f(X) = \prod_{i=1}^{n}(X - \alpha_n)$ using the product formula and deduce that

$$f'(\alpha_i) = \prod_{\substack{1 \le j \le n, \\ i \ne j}} (\alpha_i - \alpha_j).$$

(ii) Using the determinant of a Vandermonde matrix, show that

$$\begin{aligned} \Delta(1, \alpha, \ldots, \alpha^{n-1}) &= \prod_{i=1}^{n} f'(\alpha_i) \\ &= (-1)^{n(n-1)/2} \mathrm{Nm}_{K/\mathbb{Q}}(f'(\alpha)). \end{aligned}$$

(iii) Now consider the case $f(X) = X^{p-1} + X^{p-2} + \cdots + X + 1$ where p is an odd prime, and write ζ for a root of f. By writing $f(X) = \frac{X^p - 1}{X - 1}$ and differentiating using the quotient rule, prove that

$$f'(\zeta) = p\zeta^{p-1}/(\zeta - 1).$$

(iv) Deduce the formula for $\Delta(1, \zeta, \ldots, \zeta^{p-2})$ which appears in Q5(iv).

7. Let K be a number field. We say that K is **totally real** if all its embeddings are real. Show that if K is totally real then the discriminant Δ_K is positive.

8. Let ω be an algebraic integer.

(i) Show that some conjugate of ω has absolute value ≥ 1.

(ii) Suppose further that $\mathrm{Nm}(\omega) = 1$. Show that that some conjugate has absolute value ≤ 1.

(iii) (Hard!) With the help of (ii), show that $X^n + X + 3$ is irreducible over $\mathbb{Q}$ for all $n \ge 2$.

9. Let, for $n \ge 1$,

$$M_n = (1 + \sqrt{2})^n + (1 - \sqrt{2})^n.$$

Show (without expanding brackets) that $M_n \in \mathbb{Z}$, and that moreover it is the nearest integer to $(1 + \sqrt{2})^n$.