Algebraic Geometry

Problem Sheet 5

(1) (a) Let S be a finite subset of $\mathbb{A}^2$. Show that there exist two polynomials f, g such that $S = \mathbb{V}(f, g)$. (Hint: choose coordinates so that all the x coordinates of points of S are different, and choose f as a polynomial in x only.)

(b) Deduce that every finite subset of $\mathbb{P}^2$ is a set-theoretic complete intersection.

(c) Let T be a set of three non-collinear points in $\mathbb{P}^2$. Prove that the ideal of T cannot be generated by 2 elements.

(2) Let $V \subseteq \mathbb{P}^n$ be a projective algebraic set in which all components have dimension $n-1$. Prove that V is a hypersurface.

(3) Let $H \subseteq \mathbb{P}^n$ be a hyperplane. Let $V \subseteq H$ be a projective algebraic set.

Let x be a point of $\mathbb{P}^n \setminus H$. Let C be the union of all lines joining V to x (this is called the **cone** over V with vertex x).

Prove that C is a projective algebraic set and that $\dim C = \dim V + 1$. (One approach to calculating the dimension is to find a generically finite dominant rational map $V \times \mathbb{P}^1 \dashrightarrow C$.)

(4) (This question is harder than the others. I think that especially part (a) is very hard using results from the course, so you might wish to assume (a) and do the rest of the question.)

Let $V, W \subseteq \mathbb{P}^n$ be irreducible projective algebraic sets. Let $\alpha \colon \mathbb{P}^n \to \mathbb{P}^{2n+1}$, $\beta \colon \mathbb{P}^n \to \mathbb{P}^{2n+1}$ denote the maps

$$\alpha([x_0 : \cdots : x_n]) = [x_0 : \cdots : x_n : 0 : \cdots : 0],$$
$$\beta([x_0 : \cdots : x_n]) = [0 : \cdots : 0 : x_0 : \cdots : x_n].$$

Let $V' = \alpha(V)$ and $W' = \beta(W)$.

For any points $x, y \in \mathbb{P}^{2n+1}$, let L_{xy} denote the line through x and y.

Let J denote the union of all lines L_{vw} where $v \in V'$ and $w \in W'$. (This is called the **join** of V' and W'.)

(a) Prove that J is a closed subset of $\mathbb{P}^{2n+1}$.

(b) Prove that $\dim J = \dim V + \dim W + 1$. (Use a similar method to (3). You will need to use the fact that V' and W' lie in independent linear subspaces of $\mathbb{P}^{2n+1}$.)

(c) Prove that $V \cap W$ is isomorphic to the intersection between J and the hypersurfaces $Z_0 = Z_{n+1}, \ldots, Z_n = Z_{2n+1}$.

(d) Conclude that $\dim(V \cap W) \geq \dim V + \dim W - n$. In particular, if $\dim V + \dim W > n$, then $V \cap W \neq \emptyset$.