Algebraic Geometry

Problem Sheet 4

Assessed coursework – Deadline 14 March

(1) Prove that $\mathbb{P}^1 \times \mathbb{A}^1$ is not isomorphic to either an affine or a projective algebraic set.

(2) (a) Prove that $\varphi\colon [x : y : z] \mapsto [xy : yz : zx]$ defines a birational equivalence $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Write down a formula for a rational inverse ψ of φ.

(b) What are the domains of definition of φ and ψ?

(c) Write down open subsets $A, B \subseteq \mathbb{P}^2$ such that φ induces an isomorphism $A \to B$.

(3) Assume that the base field k is uncountable. Prove that a countably infinite subset of $\mathbb{A}^n$ cannot be an affine algebraic set. (Use Chevalley's theorem and induction on n.)

(4) Let $V \subseteq \mathbb{P}^n$ be a quasi-projective algebraic set and $x \in V$. Prove that there exists an open set $U \subseteq V$ which contains x and is isomorphic to an affine algebraic set. (Use the fact that the complement of a hypersurface in $\mathbb{P}^n$ is affine.)

(5) An **algebraic group** is defined to be a quasi-projective variety G together with regular maps $m\colon G \times G \to G$ (multiplication) and $i\colon G \to G$ (inverse) and a point $e \in G$ (the identity) which satisfy the usual axioms for a group. We shall write these using group notation, i.e. $gh = m(g, h)$ and $g^{-1} = i(g)$.

In this question, we shall let G be an irreducible *projective* algebraic group. The aim is to prove that the group operation on G is commutative.

Let $\varphi\colon G \times G \to G$ denote the regular map $\varphi(g, h) = ghg^{-1}h^{-1}$.

Using (4), choose an open set $U \subseteq G$ which contains e and which is *affine* (that is, U is isomorphic to an affine algebraic set).

(a) Let $S = \{h \in G : \varphi(G \times \{h\}) \subseteq U\}$. Show that $\varphi(G \times \{h\}) = \{e\}$ for all $h \in S$.

(b) Find a closed subset $Z \subseteq G \times G$ such that $p_2(Z) = G \setminus S$, where p_2 is the second projection $G \times G \to G$. Deduce that $G \setminus S$ is a closed subset of G.

(c) Show that S is non-empty.

(d) Deduce that $\varphi(G \times G) = \{e\}$.

You will need to use the completeness of projective varieties for both (a) and (b).