Algebraic Geometry

Problem Sheet 2

Assessed coursework – Deadline 16 February

Throughout, we let k be an algebraically closed field whose characteristic is not 2.

(1) Let $\varphi\colon V \to W$ be a regular map between affine algebraic sets. Prove that $\varphi^*\colon k[W] \to k[V]$ is injective if and only if the image of φ is dense in W.

(2) Prove that a regular function $\varphi\colon \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism if and only if it is given by a polynomial of degree 1.

(3) Let C denote the circle $\mathbb{V}(X^2+Y^2-1) \subseteq \mathbb{A}^2$. Let φ be the rational function $(1-Y)/X$ on C.

Determine the domain of definition of φ.

(4) Let $C = \{(x,y) \in \mathbb{A}^2 : y^2 = x^3 + x^2\}$.

Let $\varphi\colon C \dashrightarrow \mathbb{A}^1$ denote the rational map $(x,y) \mapsto y/x$.

Let $\psi\colon \mathbb{A}^1 \to C$ denote the regular map $t \mapsto (t^2-1, t^3-t)$.

We aim to show that φ is not regular at $(0,0)$.

(a) Prove that C is irreducible.

(b) Check that the image of ψ is contained in C.

(c) Find a non-empty Zariski open subset $U \subseteq \mathbb{A}^1$ such that φ is defined on $\psi(U)$.

(d) Show that $\varphi \circ \psi_{|U} = \mathrm{id}_U$.

(e) Suppose for contradiction that φ is regular at $(0,0)$. Use the fact that regular maps are continuous to show that $\varphi \circ \psi = \mathrm{id}_{\mathbb{A}^1}$.

(f) Why is it impossible to have $\varphi \circ \psi = \mathrm{id}_{\mathbb{A}^1}$?

(g) What is the domain of definition of φ?

(5) Let $f \in k[X_1, \dots, X_n]$ be any irreducible polynomial of total degree 2, that is

$$f(X_1, \dots, X_n) = \sum_{1 \le i \le j \le n} a_{ij} X_i X_j + \sum_{1 \le i \le n} b_i X_i + c$$

for some $a_{ij}, b_i, c \in k$, and at least one of the a_{ij} is non-zero.

The set $V = \mathbb{V}(f)$ for such an f is called an **irreducible quadric hypersurface**. We will show that V is birationally equivalent to $\mathbb{A}^{n-1}$.

Assume that $c = 0$ and that at least one of the b_i is non-zero. It is always possible to achieve this after a translation (you are not required to prove this).

Let p denote the origin in $\mathbb{A}^n$ and let

$$H = \{(x_1, \dots, x_n) \in \mathbb{A}^n : x_n = 1\},$$
$$H_0 = \{(x_1, \dots, x_n) \in \mathbb{A}^n : x_n = 0\}.$$

By our assumptions, we have $p \in V$ and $V \not\subseteq H_0$.

In lectures, we defined a rational map $\mathbb{A}^n \dashrightarrow H$ by projection from p onto H. Let $\pi\colon V \dashrightarrow H$ denote the restriction of this rational map to V.

(a) What is the domain of definition of π?

(b) For each point $x \in H$, let L_x denote the line through x and p. Prove that $L_x \cap V$ contains either 1 or 2 points, or else is equal to L_x.

(c) Let $U = \{x \in H : \#(L_x \cap V) = 2\}$. By writing down equations defining $H \setminus U$, show that U is a non-empty Zariski open subset of H. (In order to show that U is non-empty, it may be useful to use the fact that H is irreducible.)

(d) For $x \in U$, write $L_x \cap V = \{p, \psi(x)\}$. Find the coordinates of the point $\psi(x)$. Conclude that ψ is a rational map $H \dashrightarrow V$.

(e) Considering ψ as a rational map $H \dashrightarrow V$, what is its domain of definition $\operatorname{dom} \psi$? What is $\psi(x)$ for those points x which are in $\operatorname{dom} \psi \setminus U$?

By construction, π and ψ are rational inverses of each other and so V is birational to $H \cong \mathbb{A}^{n-1}$.