Algebraic Geometry
Problem Sheet 1

(1) Let V be the affine algebraic set in $\mathbb{A}^2$ defined by the polynomials

$$f = X^2 + Y^2 - 1, \quad g = X - 1.$$

Describe the set V and find $\mathbb{I}(V)$. Is $\mathbb{I}(V) = (f, g)$?

(2) Determine all Zariski closed subsets of the following sets:
(a) the parabola $\mathbb{V}(Y - X^2) \subseteq \mathbb{A}^2$.
(b) the union of two lines $\mathbb{V}(XY) \subseteq \mathbb{A}^2$.
(c) the circle $\mathbb{V}(X^2 + Y^2 - 1)$.
Deduce that the parabola and circle are irreducible.

(3) Describe all Zariski closed subsets of the algebraic set $\mathbb{V}(XY) \subseteq \mathbb{A}^2$.

(4) Let J be the ideal $(XY,\ XZ,\ YZ) \subseteq k[X, Y, Z]$. Describe $\mathbb{V}(J)$. What are its irreducible components? Is $\mathbb{I}(\mathbb{V}(J)) = J$?

Let $J' = (XY,\ (X - Y)Z) \subseteq k[X, Y, Z]$. Determine $\mathbb{V}(J')$ and $\operatorname{rad} J'$.

(5) Assume that the characteristic of the base field k is not 2. Find the irreducible components of the subset of $\mathbb{A}^3$ defined by the equations

$$X^2 + Y^2 + Z^2 = 0, \quad X^2 - Y^2 - Z^2 + 2 = 0.$$

(6) Decompose into irreducible components the algebraic set $V \subseteq \mathbb{A}^3$ defined by the polynomials

$$f = Y^2 - XZ, \quad g = Z^2 - Y^3.$$

(Start by factorising $Yf + g = h_1 h_2$. Then describe $V \cap \{h_1 = 0\}$ and $V \cap \{h_2 = 0\}$.)

(7) Show that the algebraic set in $\mathbb{A}^3$ defined by the equations

$$Y^2 - XZ = 0, \quad X^3 - YZ = 0$$

has two irreducible components, one of which is the set

$$C = \{(t^3, t^4, t^5) : t \in k\}.$$

Determine the other connected component.

(8) Let $V \subseteq \mathbb{A}^m$ and $W \subseteq \mathbb{A}^n$ be irreducible affine algebraic sets. Prove that the product $V \times W$ is irreducible.

Show that the Zariski topology on $\mathbb{A}^2$ is not the same as the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$ coming from the Zariski topology on $\mathbb{A}^1$ (consider the diagonal $\{(x, y) \in \mathbb{A}^2 : x = y\}$.)

(9) Prove that a hypersurface $\{\underline{x} \in \mathbb{A}^n : f(\underline{x}) = 0\}$ is irreducible if and only if f is a power of an irreducible polynomial. (You will need to use the Nullstellensatz and the fact that $k[X_1, \ldots, X_n]$ is a unique factorisation domain.)