STONE REPRESENTATION OF BOOLEAN ALGEBRAS AND BOOLEAN SPACES

MATH61000, MINI-PROJECT

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1. Introduction

Stone representation of an algebraic structure A refers to a statement roughly saying that A can be encoded in some form in a topological space X such that X is constructed out of A in a natural way. The term is coined around Marshall Stone's classical theorem from [Sto36], where this was first done for so called *Boolean algebras* (see 2.3.1 for the definition; for now it is enough to think of some algebraic structure, hence a set equipped with some operations). The text at hand is about this classical theorem, and Stone's result is proved in 3.5.

For a Boolean algebra A, the representation process associates a topological space $\mathcal{U}(A)$ to A, called *the spectrum of* A, see 3.3, from which one can reconstruct the structure A. We refer to 3.5 and the remark following it. The reconstruction process allows to analyze a Boolean algebra fully within its spectrum. This opens the possibility to think about the algebraic structure in terms of topological or even geometrical intuition. For example one can ask about the shape of the space locally (at each point) and gain systematic understanding about Boolean algebras from it. As a matter of fact one can associate many spaces with the features above to almost every algebraic structure, but these spaces contain generally much more information than the original structure and they are therefore deemed to be too complicated to be analysable or of any help. This is not the case for the spectrum of a Boolean algebra: The reason is that they have an intrinsic topological description (as *Boolean spaces*, cf. 3.1) and the representation above also goes the other way. Hence for every Boolean space there is a unique Boolean algebra from which we can reconstruct the space. This is done in 3.6.

Our two main results 3.5 and 3.6 are the backbone of the celebrated *Stone Duality* for Boolean algebras, where they are tied up in the language of category theory and the power of these theorems is made fully visible. We refer to [Kop89, Chapter 3].

The reader is assumed to have basic knowledge of general (Hausdorff) topology as can be found in [Kel75; Eng89]. Further, some acquaintance with the basic notions of partially ordered sets is required, as for example exposed in [Fuc63].

Outline of contents. In section 2 a preliminary version of the representation theorem is presented, without reference to topology. This is developed in the more general context of *distributive lattices* (see, 2.1.1 for the definition). The rational here is twofold: On the one hand, the general case is not more complicated to prove and on the other hand, Stone

Duality is also available in this more general context; the reader who wants to follow up this path will then have an adequate preparation. In section 3 we prove our main theorems 3.5 and 3.6.

In this text, the symbol $\mathbb{N}$ stands for the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, whereas $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

2. Stone representation of distributive lattices

Summary In this chapter we describe the famous representation theorem for *distributive lattices* of M. Stone, cf. [Sto37]. He proved this first for the special case of Boolean algebras in [Sto36], but in fact the proof goes through more generally.

2.1. **Distributive lattices.** We present a brief introduction to distributive lattices, suitable for our purposes. For more details we refer to [Grä11, Chapter II].

2.1.1. **Definition.** A **distributive lattice** in this text[1] is a partially ordered set $L = (L, \leq)$ with the following properties:

- **DL1** For all $a, b \in L$ the supremum of $\{a, b\}$ for the partial order $\leq$ exists. The supremum is denoted by $a \vee b$. It is also called the **join of** a **and** b.
- **DL2** For all $a, b \in L$ the infimum of $\{a, b\}$ for the partial order $\leq$ exists. The infimum is denoted by $a \wedge b$. It is also called the **meet of** a **and** b.

Hence we may view the operations $\wedge$, $\vee$ as functions $L \times L \longrightarrow L$. Notice that both operations are commutative and associative as follows immediately from their definitions; in particular, expressions of the form $a_1 \wedge \ldots \wedge a_n$ are unambiguous. However, the next requirement is not implied by the previous ones:

- **DL3** *Distributivity law for* $\wedge$ *and* $\vee$
  For all $a, b, c \in L$ we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
- **DL4** There is a smallest element for $\leq$, which we denote by $\perp$, called **bottom**. There is a largest element for $\leq$, which we denote by $\top$, called **top**.

2.1.2. *Examples.*

(i) The most common example of a distributive lattice is the powerset $\mathfrak{P}(S)$ of a set S together with the partial order given by inclusion.

(ii) There is a smallest distributive lattice, consisting of two elements $\perp < \top$. There is also a *terminal distributive lattice* consisting of exactly one element.

(iii) More generally, if L is a subset of $\mathfrak{P}(S)$ containing $\emptyset$, S and if L is closed under taking finite intersections and finite unions (in S), then $L = (L, \subseteq)$ is a distributive lattice. The operations and constants in definition 2.1.1 are given by

$$\perp = \emptyset, \ \top = S, \ a \wedge b = a \cap b, \ a \vee b = a \cup b.$$

Distributive lattices of this form are called **lattices of subsets** (of S).

(iv) The set of open subsets of a topological space is a distributive lattice. The set of closed subsets of a topological space is a distributive lattice. Both lattices are lattices of subsets of the space.

[1]In the literature, condition **DL4** is not required and the objects that we are talking about are called *bounded* distributive lattices. However we will always work under assumption **DL4** and suppress the adjective *bounded*.

(v) A distributive lattice that is not by definition a lattice of subsets is the set of propositional sentences (expressions made up of letters $p, q, r, \ldots$ using the connectives $\neg, \vee, \wedge, \Rightarrow$) modulo the equivalence relation $\sim$ saying that two such expressions have the same truth table. The order is given by $[s]_\sim \leq [t]_\sim \iff (s \Rightarrow t)$ is a tautology.

In section 2.3 below we will see another example of distributive lattices, namely *Boolean algebras*.

2.1.3. **Definition.** A map $\varphi : L \longrightarrow M$ between distributive lattices is called a **homomorphism (of lattices)** if it preserves $\perp, \top, \wedge$ and $\vee$. Explicitly, this means $\varphi(\perp_L) = \perp_M$, $\varphi(\top_L) = \top_M$, $\varphi(a \wedge_L b) = \varphi(a) \wedge_M \varphi(b)$ and $\varphi(a \vee_L b) = \varphi(a) \vee_M \varphi(b)$ for all $a, b \in L$. (For better readability we will drop the subscripts L, M of the operations when this is unambiguous.)

The homomorphism φ is called an **isomorphism (of lattices)** if it is bijective.

2.1.4. *Remark.* Let $\varphi : L \longrightarrow M$ be a homomorphism of lattices.

(i) The map φ preserves the partial orders given on L, M because $x \leq y$ is equivalent to $x = x \wedge y$ in every distributive lattice and this identity is preserved by φ.
(ii) If φ is an isomorphism, then its compositional inverse φ^{-1} is again a homomorphism: The proof is straightforward and follows tightly the lines of the proof that the compositional inverse of a bijective homomorphism of groups, is itself a homomorphism of groups.

2.2. **The representation of distributive lattices as lattices of subsets.**

2.2.1**.** We show that every distributive lattice is isomorphic to a lattice of subsets of some set S (cf. 2.2.12). The key issue is how to find S. In order to construct S we will need some preparations.

2.2.2. **Definition.** Let L be a distributive lattice. A **filter** of L is a subset F of L with the following properties.

F1 $F \neq \emptyset$.
F2 If $a, b \in F$, then $a \wedge b \in F$.
F3 If $a \in F$ and $a \leq b \in L$, then $b \in F$.[2]

Obviously L is a filter of L. A filter is **proper** if it is different from L. In virtue of F3, this is equivalent to saying that $\perp \notin F$.

2.2.3. *Examples.* Let L be a distributive lattice.

(i) Clearly L is the largest filter of L and $\{\top\}$ is the smallest filter of L.
(ii) If $a \in L$, then the set $\mathfrak{f}_a := \{b \in L \mid a \leq b\}$ is obviously the smallest filter of L containing a, called the **principal filter** of a.
(iii) If L is a lattice of subsets of a set S (cf. 2.1.2(iii)) and $p \in S$, then the set $\{a \in L \mid p \in a\}$ is obviously a proper filter of L.
(iv) In the distributive lattice L of open subsets of a topological space X, the so called *neighborhood filter* $\mathcal{N}_p = \{O \in L \mid p \in O\}$ of a point $p \in X$ is a filter of L. This is a special case of (iii)

[2]Hence by F1 we know $\top \in F$.

2.2.4. **Alternative description of filters.** *The following conditions are equivalent for every subset F of a distributive lattice L.*

(i) F is a filter.
(ii) $F \neq \emptyset$ and for all $a, b \in L$ we have

$$a \wedge b \in F \iff a \in F \text{ and } b \in F.$$

Proof. (i)⇒(ii). We know $F \neq \emptyset$ by condition **F1** in 2.2.2. The implication $\Leftarrow$ of the equivalence holds by **F2** and the implication $\Rightarrow$ follows from **F3** by noticing that $a \wedge b \leq a, b$.

(ii)⇒(i). Obviously conditions **F1** and **F2** follow from (ii). Towards **F3**, if $a \leq b \in L$ and $a \in F$, then $a \wedge b = a \in F$ and so implication $\Rightarrow$ in (ii) implies $b \in F$. □

2.2.5. **Definition.** A filter F of a distributive lattice L is called a **prime filter** if

P1 F is proper, hence $F \neq L$.
P2 For all $a, b \in L$ with $a \vee b \in F$ we have $a \in F$ or $b \in F$.

The filters in example 2.2.3(iii) are clearly prime. On the other hand, principal filters may or may not be prime. For example the principal filter of $a \in \mathfrak{P}(S)$ in example 2.1.2(i) is prime if and only if a has exactly one element.

2.2.6. **Characterization of prime filters.** *The following conditions are equivalent for every subset F of a distributive lattice L.*

(i) F is a prime filter.
(ii) $F \neq \emptyset, L$ and for all $a, b \in L$ the following equivalences hold.

$$a \wedge b \in F \iff a \in F \text{ and } b \in F$$
$$a \vee b \in F \iff a \in F \text{ or } b \in F.$$

(iii) The map

$$\chi : L \longrightarrow \{\perp, \top\},\ a \longmapsto \begin{cases} \top & \text{if } a \in F \\ \perp & \text{if } a \notin F, \end{cases}$$

is a homomorphism of lattices.

Proof. (i)⇒(ii). Since F is a proper filter we know that $F \neq L$ and by 2.2.4 we only need to show the second equivalence. The implication $\Rightarrow$ holds by **P2** and the implication $\Leftarrow$ follows from $a, b \leq a \vee b$ and **F3**.

(ii)⇒(i). By 2.2.4 we only need to show **P1** and **P2**. Since $F \neq L$ we know **P1**. The implication $\Rightarrow$ in the second equivalence of (ii) is just **P2**.

Hence we know that (i) and (ii) are equivalent.

(ii)⇔(iii). The map χ pre

serves $\perp$ and $\top$ just if $\perp \notin F$ and $\top \in F$. Hence under both assumptions (ii) and (iii) we know $\perp \notin F$ and $\top \in F$. Furthermore, the equivalences in (ii) expressed in terms of the map χ translate into

$$\chi(a \wedge b) = \top \iff \chi(a) = \top \text{ and } \chi(b) = \top$$
$$\chi(a \vee b) = \top \iff \chi(a) = \top \text{ or } \chi(b) = \top.$$

But just says that χ preserves meet and join. Thus (ii) is equivalent to (iii). $\square$

2.2.7. *Notation.* Let L be a distributive lattice. We write

$$\mathrm{PrimF}(L) = \{P \subseteq L \mid P \text{ is a prime filter}\}$$

for the set of prime filters of L. If $S \subseteq L$ we write

$$V(S) = \{P \in \mathrm{PrimF}(L) \mid S \subseteq P\}.$$

When $S = \{a\}$ with $a \in L$ we just write $V(a)$ instead of $V(\{a\})$, hence $V(a) = \{P \in \mathrm{PrimF}(L) \mid a \in P\}$. Finally we write

$$C(L) = \{V(a) \mid a \in L\}.$$

The set S promised in 2.2.1 is $\mathrm{PrimF}(L)$ and the lattice of subsets of this set, which is isomorphic to the given lattice L is supported by $C(L)$. All but one property of these statements are mere observations:

2.2.8. *Observation.* In the situation of 2.2.7 we observe the following properties.

(i) We have $V(\bot) = \emptyset$, because by **P1** no prime filter contains $\bot$. Furthermore $V(\top) = \mathrm{PrimF}(L)$ because every prime filter contains $\top$.
(ii) If $a, b \in L$ then by 2.2.6(i)⇒(ii) we know

$$V(a \wedge b) = V(a) \cap V(b), \text{ and}$$
$$V(a \vee b) = V(a) \cup V(b).$$

(iii) By (i) and (ii), the set $C(L)$ is a lattice of subsets of $\mathrm{PrimF}(L)$ and the map $\mathcal{V}_L : L \longrightarrow C(L)$ that sends $a \in L$ to $V(a)$ is a homomorphism of lattices.

Hence, once we know that the map $\mathcal{V}_L$ from 2.2.8(iii) is injective, then $\mathcal{V}_L$ is an isomorphism of distributive lattices as announced in 2.2.1. However, injectivity requires some work; notice that at the moment we even do not know whether a given distributive lattice with at least two elements possesses a prime filter.

2.2.9. **Lemma.** *Let L be a distributive lattice and let $\emptyset \neq S \subseteq L$. Then there is a smallest filter of L containing S, namely*

$$\mathfrak{f}_S = \{a \in L \mid \exists n \in \mathbb{N},\, s_1, \ldots, s_n \in S : s_1 \wedge \ldots \wedge s_n \leq a\}.$$

The filter $\mathfrak{f}_S$ is called the ***filter generated by*** *S. Notice that $\mathfrak{f}_{\{a\}} = \mathfrak{f}_a$ for $a \in L$.*

Proof. Clearly $S \subseteq \mathfrak{f}_S$. We first show that $\mathfrak{f}_S$ is a filter: Since $S \neq \emptyset$ we have $\mathfrak{f}_S \neq \emptyset$ and so **F1** of 2.2.2 holds. If $\mathfrak{f}_S \ni a \leq b \in L$ then clearly $b \in \mathfrak{f}_S$ and so **F3** holds. Now assume $a, b \in \mathfrak{f}_S$. Choose $k, n \in \mathbb{N}$ and $s_1, \ldots, s_k, t_1, \ldots, t_n \in S$ with $s_1 \wedge \ldots \wedge s_k \leq a$ and $t_1 \wedge \ldots \wedge t_n \leq b$. Then $s_1 \wedge \ldots \wedge s_k \wedge t_1 \wedge \ldots \wedge t_n \leq a \wedge b$, witnessing that $a \wedge b \in \mathfrak{f}_S$.

Hence indeed $\mathfrak{f}_S$ is a filter containing S and it remains to show that $\mathfrak{f}_S$ is contained in every filter F that contains S. Take $a \in \mathfrak{f}_S$. By definition of $\mathfrak{f}_S$ there are $n \in \mathbb{N}$ and $s_1, \ldots s_n \in S$ with $s_1 \wedge \ldots \wedge s_n \leq a$. As $S \subseteq F$, condition **F2** for F ensures $s_1 \wedge \ldots \wedge s_n \in F$. But now condition **F3** for F ensures that $a \in F$ as required. $\square$

The next proposition is central for the representation theorem 2.2.12.

2.2.10. **Proposition.** *Let F be a filter of a distributive lattice L and let $a \in L \setminus F$. Suppose that there is no proper filter G with $F \subsetneq G$ and $a \notin G$ (hence F is maximal for inclusion among filters of L not containing a). Then F is a prime filter.*

Proof. Since $a \notin F$, the filter F is proper and thus satisfy condition **P1** of 2.2.5. We need to verify condition **P2**. So take $b, c \in L$ with $b \vee c \in F$. Assume by way of contradiction that $b, c \notin F$. Let G be the filter generated by $F \cup \{b\}$ and let H be the filter generated by $F \cup \{c\}$. By the maximality assumption on F in the proposition we know that $a \in G$ and $a \in H$. By 2.2.9 there are $s_1, \ldots, s_k, t_1, \ldots, t_n \in F$ with

$$s_1 \wedge \ldots \wedge s_k \wedge b \leq a \text{ and}$$
$$t_1 \wedge \ldots \wedge t_n \wedge c \leq a.$$

Then $z := s_1 \wedge \ldots \wedge s_k \wedge t_1 \wedge \ldots \wedge t_n \in F$ by **F2** of 2.2.2 and therefore $z \wedge b, z \wedge c \leq a$. But then $(z \wedge b) \vee (z \wedge c) \leq a$ and by the distributivity law **DL3** for distributive lattices we obtain

$$z \wedge (b \vee c) = (z \wedge b) \vee (z \wedge c) \leq a.$$

However, at the beginning of the proof we have assumed that $b \vee c \in F$. Then **F2** implies $z \wedge (b \vee c) \in F$ and consequently **F3** implies $a \in F$. This contradicts the assumption of the proposition. □

2.2.11. **Corollary.** *Let F be a filter of the distributive lattice L and let $a \in L \setminus F$. Then there is a prime filter P of L containing F with $a \notin P$.*

Proof. We apply the Lemma of Zorn (cf. [Cie97, Theorem 4.3.4, p. 53]) to the set

$$\mathcal{S} = \{G \subseteq L \mid G \text{ filter of } L \text{ with } F \subseteq G \text{ and } a \notin G\}$$

furnished with the partial order $\subseteq$. If $\mathcal{C} \subseteq \mathcal{S}$ is nonempty and totally ordered for inclusion, then routine checking shows that $\bigcup \mathcal{C}$ is again a filter of L and obviously $F \subseteq \bigcup \mathcal{C}$ (as $\mathcal{C} \neq \emptyset$) and $a \notin \bigcup \mathcal{C}$. Thus $\bigcup \mathcal{C}$ is an upper bound of $\mathcal{C}$ in the partially ordered set $(\mathcal{S}, \subseteq)$. Since $\mathcal{S}$ is nonempty (it contains F) we may apply Zorn's Lemma and see that $(\mathcal{S}, \subseteq)$ has a maximal element P. By 2.2.10 we know that P is a prime filter. Since $P \in \mathcal{S}$ we obtain $F \subseteq P$ and $a \notin P$, as required. □

2.2.12**. Representation theorem for distributive lattices** (This was originally proved by Marshall Stone in [Sto37].) *Every distributive lattice L is isomorphic to the distributive lattice $C(L)$ of subsets of* $\operatorname{PrimF}(L)$ *(cf. 2.2.7). The isomorphism is given by the map $\mathcal{V}_L : L \longrightarrow C(L)$ that sends $a \in L$ to $V(a) = \{P \mid a \in P\}$.*

Proof. By 2.2.8, the only property that remains to be shown is injectivity of $\mathcal{V}_L$. Take $a, b \in L$ and assume that $b \not\leq a$. This means that a is not in the principal filter $\mathfrak{f}_b$ generated by b. By 2.2.11, there is a prime filter P of L with $\mathfrak{f}_b \subseteq P$ and $a \notin P$. Hence $P \in V(a) \neq V(b)$, which entails $V(a) \neq V(b)$ as required. □

We conclude with one notion that becomes central in the rest of the text.

2.2.13. **Definition.** A filter F of a distributive lattice L is called a **ultrafilter** if it is a maximal proper filter, i.e.,

U1 F is proper, hence $F \neq L$.
U2 If G is a filter of L with $F \subseteq G$, then $G = F$ or $G = L$.

2.2.14. *Observation.* Ultrafilters are prime by 2.2.10 applied to $a = \perp$. Furthermore ultrafilters exist in any distributive lattice that satisfies $\perp \neq \top$: apply the proof of 2.2.11 to $F = \{\top\}$ and $a = \perp$.

2.3. **Application to Boolean algebras.**

2.3.1. **Definition.** A **Boolean algebra** is a distributive lattice A that satisfies the following additional property:

BA *Existence of a complement*
For every $a \in A$ there is some $b \in A$ with $a \wedge b = \perp$ and $a \vee b = \top$.

The element b in **BA** is uniquely determined as follows from **DL3**. We may therefore define $\neg a = b$ and called it the **complement of** a (in A).

A map $\varphi : A \longrightarrow B$ between Boolean algebras is called a **homomorphism (of Boolean Algebras)** if φ is a homomorphism of lattices. By uniqueness of complements, the properties defining the complement in **BA** readily imply that $\varphi(\neg a) = \neg\varphi(a)$ for all $a \in A$, thus φ preserves complements as well.

An **isomorphism (of Boolean algebras)** is an isomorphism of distributive lattices between Boolean algebras.

2.3.2. *Example.* The prime example of a Boolean algebra is the powerset $\mathfrak{P}(S)$ of a set S, cf. 2.1.2(i). The distributive lattices in 2.1.2(iii),(iv) are in general not Boolean algebras. The distributive lattice in 2.1.2(v) is a Boolean algebra (called Tarski-Lindenbaum algebra of propositional calculus), because complements are given by $\neg[t]_\sim = [\neg t]_\sim$ for a propositional expression t.

If A is a nonempty subset of $\mathfrak{P}(S)$ that is closed under taking finite intersections and complements (in S), then $A = (A, \subseteq)$ is a Boolean algebra. The operations and constants in definitions 2.1.1 and 2.3.1 are given by

$$\perp = \emptyset,\ \top = S,\ a \wedge b = a \cap b,\ a \vee b = a \cup b^{[3]},\ \text{and } \neg a = S \setminus a.$$

Boolean algebras of this form are called **Boolean algebras of subsets** (of S). .

Obviously a Boolean algebras of subsets of S is the same as a lattice of subsets of S, which is at the same time a Boolean algebra. Hence we can write out the representation theorem 2.2.12 for Boolean algebras. However we can do slightly better, because in Boolean algebras, prime filters agree with ultrafilters:

2.3.3**. Characterization of ultrafilters in Boolean algebras.** *The following conditions are equivalent for every subset F of a Boolean algebra A.*

(i) F is an ultrafilter.
(ii) F is a prime filter.
(iii) F is a proper filter and for all $a \in A$ we have $a \in F$ or $\neg a \in F$.

[3]The choice of $a \vee b$ here makes sense, because A is closed under finite intersections and complements; now apply DeMorgan's law.

(iv) The map

$$\chi : A \longrightarrow \{\perp, \top\},\ a \longmapsto \begin{cases} \top & \text{if } a \in F \\ \perp & \text{if } a \notin F, \end{cases}$$

is a homomorphism of Boolean algebras.

Proof. (i)⇒(ii) holds by 2.2.14 in any distributive lattice.

(ii)⇒(iii) The prime filter F is proper by **P1** of 2.2.5. If $a \in A$, then $a \vee \neg a = \top \in F$ and by **P2** we get $a \in F$ or $\neg a \in F$.

(iii)⇒(i). If G is a filter of A with $F \subsetneq G$, then take $a \in G \setminus F$. Since $a \notin F$ we know that $\neg a \in F$ by (iii). But $F \subseteq G$, hence $\neg a \in G$ and as $a \in G$ we obtain $\perp = a \wedge \neg a \in G$. Thus $G = A$ as required.

Hence we know that (i), (ii) and (iii) are equivalent. However, by 2.2.6 we already know that (ii) and (iv) are equivalent (recall that every homomorphism of lattices between Boolean algebras is a homomorphisms of Boolean algebras). □

Altogether we obtain:

2.3.4. **Representation theorem for Boolean algebras I** [Sto36]
Every Boolean algebra A is isomorphic to the Boolean algebra $C(A)$ of subsets of the set of ultrafilters $\mathrm{PrimF}(A)$ *of A. The isomorphism is given by the map $\mathcal{V}_A : A \longrightarrow C(A)$ that sends $a \in A$ to $V(a) = \{P \mid a \in P\}$. Consequently $V(\neg a) = \mathrm{PrimF}(A) \setminus V(a)$ for every $a \in A$.* □

Proof. Ultrafilters are the same objects as prime filters for Boolean algebras by 2.3.3. All other assertion are immediate from 2.2.12. □

3. The representation of Boolean algebras and Boolean spaces

Summary. Theorem 2.3.4 can be considerably strengthened in topological terms, which is established in 3.5 and 3.6 below. In fact, the representation theorem 2.2.12 of distributive lattices can also be strengthened in a similar spirit as has been shown by Stone again in [Sto37]. For details we refer to [DST19, Chapter 3].

3.1. **Definition.** A **Boolean space** is a topological space X that is compact Hausdorff and such that every open set is a union of **clopen** sets (clopen means "closed and open"); in other words, the clopen sets form a basis of X.

3.2. *Remark.* Let X be a any topological space.

(i) The set $\mathrm{Clop}(X)$ of clopen subsets of X is a Boolean algebra of subsets of X, because $\emptyset, X$ are clopen and clearly intersections and complements of clopen sets are again clopen.

(ii) The most prominent Boolean space is the Cantor ternary set. One can see this directly or by invoking the following characterization: A compact Hausdorff space is Boolean if and only if it is **totally disconnected**, i.e. the only nonempty connected subsets are singletons. This is an easy consequence of [Eng89, Theorem 6.1.23], which says that every connected component of any compact Hausdorff space is the intersection of its clopen superset.

3.3. **Definition.** Let A be a Boolean algebra. We define a topological space $\mathcal{U}(A)$ associated to A, called the **spectrum of** A [4], as follows: The underlying set of $\mathcal{U}(A)$ is the set of ultrafilters of A; recall from 2.3.3 that this set is equal to the set of prime filters of A. The topology of $\mathcal{U}(A)$ is defined to be the smallest topology for which all sets of the form $V(a) = \{U \in \mathcal{U}(A) \mid a \in U\}$, $a \in A$, are closed.

3.4. **Proposition.**
The space $\mathcal{U}(A)$ *is a Boolean space and* $\mathrm{Clop}(\mathcal{U}(A)) = \{V(a) \mid a \in A\}$.

Proof. Recall that the set on right hand side was denoted by $C(A)$ in 2.2.7. By 2.3.4, the set $C(A)$ is a Boolean algebra of subsets of $\mathcal{U}(A)$ and the map $A \longrightarrow C(A)$ that sends a to $V(A)$ is an isomorphism of Boolean algebras. It follows that the set of all intersections of sets of the form $V(a)$ is the set of closed sets of a topology on $\mathcal{U}(A)$ and consequently this has to be the topology defined in 3.3. Consequently,

$(*)$ every closed set of $\mathcal{U}(A)$ is an intersection of sets of the form $V(a)$ with $a \in A$.

Claim 1. The space $\mathcal{U}(A)$ is compact.
Proof of claim 1. By virtue of property $(*)$, it suffices to show that every subset $\mathcal{S}$ of $C(A)$ with the property that every finite subset of $\mathcal{S}$ has nonempty intersection (this property of a set of subsets of a given set is referred to as **finite intersection property**), has nonempty intersection.

Let $S = \{a \in A \mid V(a) \in \mathcal{S}\}$. We first show that F is proper. Otherwise $\perp \in F$ and by 2.2.9 there are $a_1, \ldots, a_n \in S$ with $a_1 \wedge \ldots \wedge a_n = \perp$. But then $\emptyset = V(\perp) = V(a_1 \wedge \ldots \wedge a_n) = V(a_1) \cap \ldots \cap V(a_n)$; since all $V(a_i)$ are in $\mathcal{S}$, this contradicts the finite intersection property. Hence F indeed is a proper filter and by 2.2.11 there is a prime filter U of A containing F. By 2.3.3 we know $U \in \mathcal{U}(A)$ and we show that $U \in \bigcap \mathcal{S}$: Take $S \in \mathcal{S}$. Then $S = V(a)$ for some $a \in F$ by choice of F. Since $F \subseteq U$ we get $U \in V(a)$ as required. $\diamond$

Claim 2. $\mathrm{Clop}(\mathcal{U}(A)) = \{V(a) \mid a \in A\}$.
Proof of claim 2. $\supseteq$: Take $a \in A$. Since $V(\neg a) = \mathcal{U}(A) \setminus V(a)$, the set $V(a)$ is open. It is closed by definition of the topology, hence $V(A) \in \mathrm{Clop}(\mathcal{U}(A))$.

$\subseteq$. Let $\mathcal{K} \subseteq \mathcal{U}(A)$ be clopen. Since $\mathcal{K}$ is open we know from $(*)$ that the complement of $\mathcal{K}$ is an intersection of sets from $C(A)$. By taking complements and takeing into account that $V(a)$ has complement $V(\neg a)$ for $a \in A$ we see that $\mathcal{K}$ is a union of sets of the form $V(b)$ with $b \in A$. As $\mathcal{K}$ is also closed it is compact, using claim 1. It follows that $\mathcal{K}$ is a finite union of sets of the form $V(b)$ with $b \in A$. Hence there are $b_1, \ldots, b_n \in A$ with $\mathcal{K} = V(b_1) \cup \ldots \cup V(b_n)$. However, the latter set is equal to $V(b_1 \vee \ldots \vee b_n)$, which is in $C(A)$. $\diamond$

Claim 2 together with property $(*)$ also implies that every open set is a union of sets from $C(A)$ and so $\mathcal{U}(A)$ is Boolean. It remains to show that $\mathcal{U}(A)$ is Hausdorff. So take $U_1, U_2 \in \mathcal{U}(A)$ with $U_1 \neq U_2$. Without loss of generality we may assume that there is some $a \in U_1 \setminus U_2$. By 2.3.3 we know $\neg a \in U_2$. Hence $U_1 \in V(a)$, $U_2 \in V(\neg a)$ and $V(a) \cap V(\neg a) = V(a \wedge \neg a) = V(\perp) = \emptyset$. Since $V(a)$ and $V(\neg a)$ are open, this implies that $\mathcal{U}(A)$ is Hausdorff. $\square$

[4] In the literature, $\mathcal{U}(A)$ is also called the **Stone space**, cf. [Joh86, II 4.2, bottom of p. 70] or **space of ultrafilters** of A.

We can now improve 2.3.4 by invoking 3.4 to obtain

3.5. **Representation theorem for Boolean algebras II** (This was originally proved by Marshall Stone in [Sto36].) *Every Boolean algebra A is isomorphic to the Boolean algebra* $\mathrm{Clop}(\mathcal{U}(A))$ *of the Boolean space* $\mathcal{U}(A)$.

The isomorphism is given by the map $\mathcal{V}_A : A \longrightarrow \mathrm{Clop}(\mathcal{U}(A))$ *that sends* $a \in A$ *to* $V(a) = \{P \mid a \in P\}$. □

Theorem 3.5 says something remarkable: Given a Boolean algebra A we have constructed the topological space $\mathcal{U}(A)$. Now using 3.5 we see that we can reconstruct A (up to isomorphism) from this topological space. The mechanism also works in the opposite direction:

3.6. **Representation theorem for Boolean spaces**
Let X be a Boolean space. Then $\mathrm{Clop}(X)$ *is a Boolean algebra of subsets of X and the map* $\Theta_X : X \longrightarrow \mathcal{U}(\mathrm{Clop}(X))$ *defined by* $\Theta_X(x) = \{K \in \mathrm{Clop}(X) \mid x \in K\}$ *is a homeomorphism.*

The compositional inverse is given as follows: If $U \in \mathcal{U}(\mathrm{Clop}(X))$, *then the intersection* $\bigcap U$ *has exactly one element and this element is* $\Theta_X^{-1}(U)$.

Proof. Firstly we observe that the map Θ_X is indeed well defined, i.e., for $x \in X$ the set $\Theta_X(x)$ is an ultrafilter, also see example 2.2.3(iii). For the rest of the proof we suppress the index X from Θ_X and just write Θ. The essential part of the assertion is the following

Claim. For each $U \in \mathcal{U}(\mathrm{Clop}(X))$ there is some $x \in X$ with $\bigcap U = \{x\}$. We write $\Psi(U)$ for this element and obtain a map $\Psi : \mathcal{U}(\mathrm{Clop}(X)) \longrightarrow X$.
Proof of the claim. Since U is a proper filter, U has the finite intersection property. Since all elements of U are closed sets and X is compact, we know that $\bigcap U \neq \emptyset$. We need to show that there is at most one point in $\bigcap U$. Suppose for way of contradiction that there are two points $x, y \in \bigcap U$. Since $x \neq y$ and X is Hausdorff, there are open and disjoint neighborhoods O, W of x, y respectively. Since X is Boolean there are clopen subsets K, L of X with $x \in K \subseteq O$ and $y \in L \subseteq W$. From $O \cap W = \emptyset$ we get $K \cap L = \emptyset$ and therefore $(X \setminus K) \cup (X \setminus L) = X$. Since U is a filter we know $X \in U$. However, $X \setminus K$ and $X \setminus L$ are in the Boolean algebra $\mathrm{Clop}(X)$ and so the ultrafilter property of U implies $X \setminus K \in U$ or $X \setminus L \in U$. By symmetry we may assume that $X \setminus L \in U$. But then $\bigcap U \subseteq X \setminus L$ and this contradicts $y \in \bigcap U \cap L$, establishing the claim. ⋄

We now proof that Ψ is the compositional inverse of Θ. For $x \in X$ we have $x \in \bigcap \Theta(x)$ by definition of $\Theta(x)$ and so by the claim this implies $\Psi(\Theta(x)) = x$. Thus $\Psi \circ \Theta = \mathrm{id}_X$. Further, if $U \in \mathcal{U}(\mathrm{Clop}(X))$ we have $\Theta(\Psi(U)) = \{K \in \mathrm{Clop}(X) \mid \Psi(U) \in K\} \supseteq U$ by definition of $\Psi(U)$; since both U and $\Theta(\Psi(U))$ are ultrafilters we get $\Theta(\Psi(U)) = U$. This shows $\Theta \circ \Psi = \mathrm{id}_{\mathcal{U}(\mathrm{Clop}(X))}$ and so indeed Ψ is the compositional inverse of Θ.

Finally we need to show that Θ is a homeomorphism. It is continuous because for $K \in \mathrm{Clop}(X)$ we have

$$\begin{aligned}
\Theta^{-1}(V(K)) &= \{x \in X \mid \Theta(x) \in V(K)\} \\
&= \{x \in X \mid K \in \Theta(x)\}, \text{ by definition of } V(K) \\
&= \{x \in X \mid x \in K\}, \text{ by definition of } \Theta(x) \\
&= K,
\end{aligned}$$

which is closed, and because every closed sets of $\mathcal{U}(\mathrm{Clop}(X))$ is an intersection of sets of the form $V(K)$ with $K \in \mathrm{Clop}(X)$ (see property $(*)$ in the proof of 3.4).

Hence we know that Θ is a continuous bijective between the compact Hausdorff spaces X and $\mathcal{U}(\mathrm{Clop}(X))$ (invoke 3.4) and every such map is a homeomorphism by general topology. $\square$

As indicated at the beginning of section 3, both representation theorems 3.5 and 3.6 can be generalised to all distributive lattices. On the topological side one has to switch from Boolean spaces to so called *spectral spaces*. This can be found in various formulations in [DST19, Section 3.2].

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