SIMPLE GROUPS AND THE JORDAN-HOLDER THEOREM

MATH61000 MINI-PROJECT

XXXXXX XXXXXX

Student ID: XXXXXXX

November 3, 2020

Department of Mathematics, The University of Manchester

Contents

1. Introduction

1.1. **Abstract.** Our main motivation in this project is to introduce simple groups and to prove the Jordan-Holder Theorem. The Jordan-Holder Theorem states that any two composition series (4.1.2) for a finite group G have the same length and the same set of composition factors. This tells us that for each finite group G we can associate a unique set of simple groups. [Ros09] refers to the composition factors as the 'building blocks' of a group G. An important application of the theorem is that it motivates 'Holder's Program' [Mac12] for the classification of finite groups, which says for groups with composition series, if we can

(i) list all finite simple groups,
(ii) solve the extension problem: given two groups K and H, determine all groups G which contain a normal subgroup which is isomorphic to K and such that $H \simeq G/K$. We say G is an extension of K by H.

then we can describe all groups. Essentially this tells us we can use the Jordan Holder Theorem to construct all finite and even some infinite groups from simple groups using extensions. This indicates the importance of the concept of a simple group when trying to achieve a complete picture of all groups. Following the reading of this project we direct the reader to [Ros09, Extension Problem 9.2] for further discussion of this application.

Before we approach the Jordan-Holder Theorem statement and proof we will recall some material involving normal subgroups and factor groups. We will also introduce simple groups and prove another important theorem that A_n is simple for $n \geq 5$. These chapters describe key mathematical concepts and contain fundamental technical material for the Jordan-Holder Theorem statement and proof.

The intended audience for this project is a mathematics undergraduate who has taken an introductory algebra course and is familiar with groups, subgroups, normal groups, conjugacy and the isomorphism theorems. We direct the reader to [Kna06] if a revision of the prerequisite material is required.

1.2. **Outline of Contents.** In section 2 we will recall the definition of a normal subgroup and prove some results that we will need in section 4 for our proof of the Jordan-Holder Theorem. We will also recall the definition of a factor group G/H, as well as stating some important lemmas relating to factor groups.

Section 3 will introduce simple groups, providing a definition and detailing some key examples. We will also present the important proof that A_n is simple for $n \geq 5$.

In the fourth and final section we will state and prove the Jordan-Holder Theorem. We will first define a composition series and provide examples as motivation for our proof. We will then give two proofs of the Jordan Holder Theorem, one by induction and one using the Zassenhaus Lemma and the Schreier Refinement Theorem.

1.3. **Acknowledgement of Referenced Material.** A list of all referenced material used in this project can be found in the bibliography. Referenced text is identified by the use of quotation marks and the mentioning of the source in the text. Definitions and Theorems provided in this project will come from the referenced sources unless stated otherwise. The proofs presented in this project are adapted from the referenced sources, reworded and in some cases developed further. These are referenced accordingly.

1.4. **Preliminaries and Notation.** In this text we will use the following notation:

- We will denote the order of a group G by $\mid G \mid$.
- We use the notation $H \leq G$ to say H is a subgroup of a group G, and $H < G$ if H is a proper subgroup of G.
- We will denote the trivial group, the group containing only the identity element, by $\{1\}$.
- We denote the symmetric group by S_n and alternating group of all the even permutations of S_n by A_n.
- We will use gH and Hg to denote the set of left and right cosets of a subgroup H respectively.

2. Normal Subgroups & Factor Groups

Summary. In this section we will recall what it means for a subgroup H of G to be normal in G and for G/H to be a factor group. We will give some examples and prove some important results which will we need in section 4 for our proof of the Jordan-Holder Theorem.

2.1. **Normal Subgroups.** The definitions and theorems in this subsection are adapted from [Ros09, Section 2.4] unless otherwise stated.

2.1.1. **Definition.** Let $N \leq G$. N is **normal** in G if and only if $gN = Ng$, for all $g \in G$. That is, if and only if the left and right cosets of N are equal. If N is a normal subgroup of G we denote this by $N \trianglelefteq G$.

2.1.2. **Theorem.** *Let G be a group, $N \leq G$. Then the following are equivalent:*

(i) $N \trianglelefteq G$
(ii) for all $g \in G$, $g^{-1}Ng \subseteq N$
(iii) for all $g \in G$, *for all* $n \in N$, $g^{-1}ng \in N$

Our proof that these conditions are equivalent follows that of [Ros09, Theorem 2.29]

Proof. (i) $\implies$ (ii):
Suppose $N \trianglelefteq G$ and let $g \in G$. Then from our definition of a normal subgroup, $gN = Ng$. Then for all $n \in N$ there exists $n' \in N$ such that $gn' = ng$. Applying g^{-1} on the left of both sides of our equation gives $n' = g^{-1}ng \in N$ as required.

(ii) $\implies$ (iii):
This is immediate as $g^{-1}ng \in g^{-1}Ng$ for all $n \in N$.

(iii) $\implies$ (i):
Let $g \in G, n \in N$. Then there exists $n' \in N$ satisfying $g^{-1}ng = n'$ which implies $ng = gn'$. So $Ng \subseteq gN$. Conversely, $gng^{-1} = (g^{-1})^{-1}n(g^{-1}) \in N$ so we can find $n'' \in N$ which satisfies $gng^{-1} = n''$. Equivalently $gn = n''g$, so $gN \subseteq Ng$. So we have shown $gN = Ng$ and so by definition N is a normal subgroup of G. □

2.1.3. *Examples.*

(i) In a group G, the trivial group and G itself are normal.
(i) For an abelian group G, all subgroups are normal as all elements g are conjugate to themselves so the conjugacy class of g is g.

2.1.4. **Theorem.** *If* $H \trianglelefteq G$ *and* $J \trianglelefteq G$, $HJ \trianglelefteq G$.

We use [Ros09, THM 2.30] as the source of our proof.

Proof. Recall $HJ \leq G$ for $H \leq G$, $J \leq G$. So we just need to show HJ is normal in G. Let $g \in G, h \in H$ and $j \in J$. Now $g^{-1}hjg = g^{-1}hgg^{-1}jg \in HJ$. Since H and J are normal $g^{-1}hg \in H$ and $g^{-1}jg \in J$ so $HJ \trianglelefteq G$ by 2.1.2(iii). □

Following this we can state and proof another lemma.

2.1.5. **Lemma.** *Let H, K be groups. If $J \trianglelefteq H$, then $JK \trianglelefteq HK$.*

Proof. Since $J \leq H$, recall $JK \leq HK$. We then want to show normality. Let $j \in J$, $k \in K$. Since $j \in H$, $g^{-1}jg \in HJ$ so $JK \trianglelefteq HK$. □

2.1.6. **Lemma.** *Let $H \trianglelefteq G$, $K \trianglelefteq G$. Then $H \cap K \trianglelefteq G$*

Proof. Recall that for $H \leq G$, $K \leq G$, $H \cap K \leq G$. So we just need to prove normality of $H \cap K$. Since $K \trianglelefteq G$, $h^{-1}jh \in K$ if $j \in K$ by 2.1.2. Now let $h \in H$. If $j \in H$, $h^{-1}jh \in H$. Therefore if $j \in H \cap K$, $h^{-1}jh \in H \cap K$. □

A proper normal subgroup N of G is said to be maximal if G has no proper normal subgroups larger than N. We give the definition of a maximal normal subgroup as given in [Mac12].

2.1.7. **Definition.** A normal subgroup N is said to be a **maximal normal subgroup** if $N \leq K$ and $K \trianglelefteq G$, then either $N = K$ or $K = G$.

2.2. **Factor Groups.** The definitions and proofs in this subsection follow [KM77, Chapter 2] unless otherwise stated.

2.2.1. **Definition.** We let G/H denote the set of all the left cosets of H in G. Recall that for $g \in G$, $gH = \{gh \mid h \in H\}$ is a left coset of H in G. When H is normal we say G/H is the **factor group** or **quotient group** of G, with group multiplication defined by $(g_1H)(g_2H) = (g_1g_2H)$.

Recall that the index of H in G denoted [G:H] is the number of cosets of H in G. We will follow the idea of [Mac12, Example 2.2.2] to prove that if the index of H in G is equal to 2, H is normal.

2.2.2. **Lemma.** *If* $[G : H] = 2$*, then* $H \trianglelefteq G$*.*

Proof. Let G be a group, $H \leq G$. Let $g \in G$. We want to show that $Hg = gH$. If $g \in H$ then $Hg = H = gH$ and then we are done. Suppose $g \notin G$. Then $G = H \cup Hg = H \cup gH$, which is a disjoint union. So $G/H = Hg = gH$ and $gH = Hg$. So $H \trianglelefteq G$. □

2.2.3. *Examples.* $[S_n : A_n] = 2$ so A_n is normal in S_n

We recall Lagrange's Theorem here as it will be useful in section 4 when we are constructing a composition series for G. However, we will not prove it here as it is expected prerequisite material.

2.2.4. **Theorem.** *(Lagrange's Theorem)*
Let G be a finite group, $H \leq G$*. Then* $|G| = |G/H| \cdot |H|$

In our proof of the Jordan Holder Theorem in Section 4 we use the Second Isomorphism Theorem. The reader is expected to be familiar with it so we will state the theorem here but not present the proof. Our statements come from [Ros09] and we direct the reader to [Ros09, Theorem 4.14] for the proof.

2.2.5. **Theorem.** *(The Second Isomorphism Theorem)*
Let $H \leq G$ *and* $N \trianglelefteq G$*. Then,*

(i) $N \trianglelefteq NH \leq G$*, and*
(ii) $H/(H \cap N) \simeq NH/N$

3. Simple Groups

Summary. In this section we define what it means for a group G to be simple from [Kna06]. We introduce some examples of simple groups and prove two lemmas relating maximal normal subgroups and simple groups. We then present the important proof that A_n is simple for $n \geq 5$.

3.0.1. **Definition.** A group G is **simple** if and only if its only normal subgroups are itself and $\{1\}$.

3.0.2. *Examples.*

(1) $\mathbb{Z}_p$, the cyclic group of prime order is simple by Lagrange's Theorem 2.2.4 since the order of any subgroup H of G has to divide the order of G, so its only subgroups are G and $\{1\}$ which are both normal in $\mathbb{Z}_p$.
(2) $\mathbb{Z}$ isn't simple as $2\mathbb{Z}$ is a non-trivial, proper normal subgroup.

Now we state and prove two lemmas involving maximal normal subgroups. The first statement and proof follows [Mac12, Lemma 2.2] and the second [Ros09, Lemma 9.2].

3.0.3. **Lemma.** *If N is a maximal normal subgroup of G, then G/H is simple.*

Proof. If $K/H \trianglelefteq G/H$ then $K \trianglelefteq G$ by the Third Isomorphism Theorem. Then if $H \lneqq K$ we have $K = G$ and the factor group K/H is the whole group or $H = K$ and $K/H = H$ is the identity. So G/H is simple. □

3.0.4. **Lemma.** *Let $H \trianglelefteq G$, $K \trianglelefteq G$. Let H and K both be maximal subgroups but $H \neq K$. Then $H \cap K$ is a maximal normal subgroup of H and K.*

Proof. Recall that since $H \trianglelefteq G$, $K \trianglelefteq G$, $H \trianglelefteq HK \trianglelefteq G$ by 2.1.4. By the maximality of H, either $H = HK$ or $HK = G$. If $H = HK$ then $K < H$ and since $H \neq K$ and $K \leq HK$ we get a contradiction to the maximality of K. So HK must equal G and by 2.2.5 we have $G/K = HK/K \simeq H/(H \cap K)$. G/K is simple by 3.0.3 and by similar argument we see $H/(H \cap K)$ is simple and so $H \cap K$ is a maximal normal subgroup of H. We argue similarly to show $H \cap K$ is a maximal normal subgroup of K and our result follows. □

We now prove that A_n is simple for $n \geq 5$. We separate our proof into lemmas. We want to show that A_n has no normal subgroups other that itself and the trivial group. We suppose for contradiction that there is a normal subgroup N of A_n but we will show N must be equal to A_n.

3.0.5. *Remark.* $A_1 = A_2 = (1)$ and A_3 is a group of prime order 3 so A_n is simple for $n = 1, 2, 3$. A_4 is not simple. For example $H = \{(1), (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4.

First we prove that every element of A_n for $n \geq 3$ can be written as the product of three cycles. We say A_n is generated by its 3 cycles for $n \geq 3$. We follow the proof given by [Ros09, Theorem 3.12].

3.0.6. **Lemma.** *A_n is generated by three cycles for $n \geq 3$*

Proof. We want to show that the product of two transpositions (two cycles of length two) can be written as the product of three cycles. Consider $1 \leq \alpha, \beta, \gamma, \delta \leq n$ where $\alpha, \beta, \gamma, \delta$ are all distinct. Then we have the following cases for the product of two transpositions:

(i) $(\alpha, \beta)(\alpha, \beta)$=(1)

(ii) $(\alpha, \beta)(\alpha, \gamma) = (\alpha, \gamma, \beta)$

(iii) $(\alpha, \beta)(\gamma, \delta) = (\alpha, \beta)(\alpha, \gamma)(\gamma, \alpha)(\gamma, \delta) = (\alpha, \gamma, \beta)(\gamma, \delta, \alpha)$ (*using case (ii) and inserting the identity*)

We have shown every pair of transpositions is either the identity or a product of three cycles.

Now take $\sigma \in A_n$, then $\sigma = \tau_1\tau_2....\tau_k$, this is a product of transpositions for some even k. We can pair these transpositions and let $\sigma = (\tau_1, \tau_2)(\tau_3, \tau_4)....(\tau_{k-1}, \tau_k)$. Then each pair is one of the cases we have presented in beginning of this proof so σ is a product of three cycles. □

Now we adapt the proof of [KM77, Theorem 12.1] to prove the following lemma.

3.0.7. **Lemma.** *Any three-cycles are conjugate in A_n for $n \geq 5$, so if $N \trianglelefteq A_n$ contains a three cycle, it contains all three cycles.*

Proof. Let $N \trianglelefteq A_n$ and take an element $\sigma = (a_1, a_2, a_3) \in N$. Let $\gamma = (b_1, b_2, b_3) \in A_n$. Let $\delta \in S_n$ such that $\delta(a_i) = b_i$ for $i = 1, 2, 3$ and $\delta\sigma\delta^{-1} = \gamma$ since three cycles are conjugate in S_n. If $\delta \in A_n$ we are done. Suppose $\delta \in S_n \setminus A_n$. Since $n \geq 5$ we can choose elements $x_1, x_2 \in \Sigma \setminus \{a_1, a_2, a_3\}$. Let $\mu = (x_1, x_2)$. Let $\delta(x_1) = y_1$ and $\delta(x_2) = y_2$. Clearly $y_1, y_2 \notin b_1, b_2, b_3$. Since σ and μ are disjoint $\sigma\mu = \mu\sigma$. Then $(\mu\delta)^{-1}\sigma(\mu\delta) = \delta^{-1}\mu^{-1}\sigma\mu\delta = \delta^{-1}\sigma\delta = \gamma$. We have shown least one of δ and $\mu\delta$ must be an even permutation. So we have shown any three cycles are conjugate in A_n. □

We now give the following definition and lemma from [SS15, Section 5.2].

3.0.8. **Definition.** Let G be a group. We define a **commutator** in G to be an element of the form $x^{-1}y^{-1}xy = [x, y]$.

3.0.9. **Lemma.** *Let G be a group, $N \trianglelefteq G$. Let $g \in G$ and $n \in N$. Then $g^{-1}n^{-1}gn \in N$.*

Proof. We know $g^{-1}n^{-1}g \in N$ as $N \trianglelefteq G$ and $n^{-1} \in N$ as N is a group. Since $n \in N$ as well, it follows $g^{-1}n^{-1}gn \in N$ □

For the proof of Lemma 3.0.10 and Theorem 3.0.11 we will adapt the proof given in [KM77, Theorem 12.1].

3.0.10. **Lemma.** *Let $N \trianglelefteq A_n$. N contains at least one three cycle.*

Proof. Let $\{1\} \neq N \trianglelefteq A_n$ with $n \geq 5$. Let $\sigma \in N$, $\sigma \neq (1)$. Then the representation of σ as a product of disjoint cycles must take one of the following forms:

(i) σ has at least one cycle of length at least four

(ii) σ has at least one cycle of length 3 and another of length at least 2

(iii) σ is a three cycle

(iv) σ is the product of two disjoint transpositions.

We choose the symbols 1,2,..,5 for convenience and can express σ as follows:

(i) $\sigma = (1234\ldots)$

(ii) $\sigma = (123\ldots)(12\ldots)$
(iii) $\sigma = (123)$
(iv) $\sigma = (12)(34)$

If σ is as in case (iii) the lemma is proven. For the other cases we will use 3.0.9 which says $[\sigma, \rho] = \sigma^{-1}\rho^{-1}\sigma\rho \in N$.

In case (i) let $\rho = (123)$. Then $[\sigma, \rho] = (1234\ldots)^{-1}(123)^{-1}(1234\ldots)(123) = (234)$ which is a three cycle.

In case (ii) let $\rho = (124)$. Then $[\sigma, \rho] = ((123)(45\ldots))^{-1}(124)^{-1}(123)(45\ldots)(124)$ $= (12435)$. From case (i) this cycle of length five can be expressed as a three cycle.

In case (iv) let $\rho = (123)$ $[\sigma, \rho] = ((12)(34))^{-1}(123)^{-1}(12)(34)(123) = (14)(23)$. Since (12435) is even and $(14)(32) \in N, (12435)^{-1}(14)(23)(12435) = (14)(25)$. Then $(14)(23)(14)(25) = (253)$.

So we have shown that N contains at least one three cycle. □

3.0.11. **Theorem.** *A_n is simple for $n \geq 5$*

Proof. We have shown in 3.0.10 that a normal subgroup N of A_n for $n \geq 5$ contains at least one three cycle. So by 3.0.7 since all three cycles are conjugate in A_n, all three cycles are in N. Therefore by 3.0.6 N contains all the elements of A_n and so $N = A_n$. We has shown A_n has no proper normal subgroups and so is simple. □

4. The Jordan-Holder Theorem

Summary. In this section we state and prove the Jordan-Holder Theorem. First we give some definitions required for the statement of the Theorem and state some preliminary lemmas. We then give a proof of the Jordan-Holder Theorem using induction. Then we prove the Zassenhaus Lemma and Schreier Refinement Theorem and present an alternative proof of the Jordan-Holder Theorem using these.

4.1. **The Jordan-Holder Theorem.** Firstly we define a normal series and composition series of G as given in [Kna06, Chapter 4, Section 8].

4.1.1. **Definition.** Let G be a group. The sequence $G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots\ G_{n-1} \trianglerighteq G_n$ of subgroups of G, where our notation means each G_k is normal in G_{k-1}, and $G_n = \{1\}$, is called a **normal series** of G.

4.1.2. **Definition.** When a normal series for G is such that every factor group G_{k-1}/G_k is simple and every inclusion $G_k \subseteq G_{k-1}$ is proper, we say that the sequence is a **composition series**. We call the G_{k-1}/G_k the **composition factors** of G.

4.1.3. *Remark.* Recall $|G/H| = [G : H]$ by definition and $[G : G_1] = \frac{|G|}{|G_1|}$ by Lagrange's Theorem 2.2.4.

Now we give some illustrative examples.

4.1.4. *Examples.*

(i) Let $G = S_4$. Since we know $A_4 \leq S_4$ and $[S_4 : A_4] = 2$, by 2.2.2, $A_4 \trianglelefteq S_4$. So for G_1 we choose A_4. $|G/G_1| \ = [G : G_1] = \frac{|G|}{|G_1|} = \frac{24}{12} = 2$. Therefore $G/G_1 \simeq \mathbb{Z}_2$ and we know $\mathbb{Z}_2$ is simple as it is a cyclic group of prime order. $< (12)(34), (13)(24) >$ is normal in A_4 so we take this as our G_2. $|G_1/G_2| \ = [G_1 : G_2] == \frac{12}{4} = 3$ so $G_1/G_2 \simeq \mathbb{Z}_3$. Since G_2 is an abelian group, all its subgroups are normal. Therefore we have three choices for G_3. We choose $G_3 =< (12)(34) >$. Then $|G_2/G_3| = 2$ and so $G_2/G_3 \simeq \mathbb{Z}_2$. Our only choice for G_4 is the trivial group. So a composition series for S_4 is $S_4 \trianglerighteq A_4 \trianglerighteq < (12)(34), (13)(24) > \trianglerighteq < (12)(34) > \trianglerighteq \{1\}$ with composition factors $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2$.

(ii) Let $G = S_n$ for some $n \geq 5$. By 2.2.3, $[S_n : A_n] = 2$ and so $A_n \trianglelefteq S_n$ by 2.2.2. $|S_n/A_n| = 2$ and so $S_n/A_n \simeq \mathbb{Z}_2$. Then since A_n is simple, the only choice for G_2 is $\{1\}$ and $G_1/G_2 \simeq A_n$. $S_n \trianglerighteq A_n \trianglerighteq \{1\}$ is then a composition series for G since $S_n/A_n \cong \mathbb{Z}_2$ and $A_n/\{1\} \cong A_n$ and $\mathbb{Z}_2$ and A_n are simple.

(iii) Let $G = \mathbb{Z}_6$. $\mathbb{Z}_6 \trianglerighteq \langle 2 \rangle \trianglerighteq \{0\}$ is a composition series for $\mathbb{Z}_6$ with $G/G_1 \cong \mathbb{Z}_2$ and $G_1/G_2 \cong \mathbb{Z}_3$. $\mathbb{Z}_6 \trianglerighteq \langle 3 \rangle \trianglerighteq \{0\}$ is a also composition series for $\mathbb{Z}_6$ with $G/G_1 \cong \mathbb{Z}_3$ and $G_1/G_2 \cong \mathbb{Z}_2$.

4.1.5. *Remark.* From 4.1.4(iii) we can see that the composition series for a group G is not unique as we have given two possible composition series for G. Note that the composition factors are the same in both composition series.

Using lemma 3.0.3, we adapt the comment made in [Mac12] following the proof of Lemma 2.2 into a proof that every finite group G has at least one composition series.

4.1.6. **Theorem.** *Let G be a finite group. Then it has at least one composition series.*

Proof. Let G be a finite group. If G itself is simple it has composition series $G \trianglerighteq \{1\}$. Suppose G is not simple and not trivial. Choose a maximal normal subgroup N_1 of G. Let $N_1 = G_1$ in our composition series. By 3.0.3 G/G_1 is simple. We continue in this way, choosing N_2 a maximal normal subgroup of N_1, until we get $G_r = \{1\}$. We have then constructed a composition series for G. □

We use definition of equivalence given by [Ros09, Definition 9.4] to form our statement of the Jordan Holder Theorem.

4.1.7. **Definition.** Let $G \neq \{1\}$ be a group and let $G \trianglerighteq H_1 \trianglerighteq ... \ H_{r-1} \trianglerighteq H_r$ and $G \trianglerighteq K_1 \trianglerighteq ... \ K_{s-1} \trianglerighteq G_s$ be two normal series of G. We say these are **equivalent** if $r = s$ and the composition factors of the two composition series are the same up to isomorphism.

4.1.8. **Theorem.** *(The Jordan-Holder Theorem)*
Suppose G is a finite group. Then all composition series for G are equivalent.

In our proof by induction we follow the proof given by [Ros09, Theorem 9.5].

Proof. We prove this theorem using induction on the order of G. If G is the trivial group, we have nothing to prove. If G is simple it's only proper normal subgroup is the trivial group so $G \trianglerighteq G_1 = \{1\}$ is the only composition series of G and we are done. So we will assume that G is not trivial and not simple.

For our inductive hypothesis we suppose that for a finite group G of order n, all the composition series of G are equivalent. From this we can assume that the result holds for all groups G' of order less than G.

Now suppose $G \trianglerighteq H_1 \trianglerighteq ...H_{r-1} \trianglerighteq H_r = 1$ and $G \trianglerighteq K_1 \trianglerighteq ...K_{s-1} \trianglerighteq K_s = 1$ are two composition series for G. We have the following two cases:

Case 1: $H_1 = K_1$
If $H_1 = K_1$ we can use our inductive hypothesis as the order of $H_1 = K_1$ is less than G since $H_1 = K_1$ is a proper subgroup of G. We conclude the two composition series are therefore equivalent.

Case 2: $H_1 \neq K_1$
For this case we will use the facts that $H_1 \cap K_1 \trianglelefteq H_1$ and $H_1 \cap K_1 \trianglelefteq K_1$ from 3.0.4. We let $L = H_1 \cap K_1$. If $L = \{1\}$ we have two equivalent composition series so suppose $L \neq \{1\}$. Now let $L = H_1 \cap K_1 \trianglerighteq L_1 \trianglerighteq ... \trianglerighteq L_t = 1$ be a composition series for L. This series exists and is non-trivial by 4.1.6.

Recall from 2.1.4 that $H_1K_1 \trianglelefteq G$ since $H_1 \trianglelefteq G$ and $K_1 \trianglelefteq G$.

Also recall since $H_1 \leq H_1K_1$ and $K_1 \leq H_1K_1$ and we cannot have H_1 or K_1 equal to H_1K_1 otherwise $H_1 = K_1$. So we suppose without loss of generality $H_1 \lneq H_1K_1$ $\trianglelefteq G$. By 3.0.3, G/H_1 is simple so H_1 is maximal. Therefore $G = H_1K_1$.

Now we use the Second Isomorphism Theorem 2.2.5(ii) which tells us $G/H_1 = H_1K_1/H_1 \simeq K/K_1 \cap H_1 = K_1/L$ and $G/K_1 = H_1K_1/K_1 \simeq H/H_1 \cap K_1 = H_1/L$.

We now have four composition series for G:

(i) $G \trianglerighteq H_1 \trianglerighteq H_2 \trianglerighteq ... \trianglerighteq H_r = 1$
(ii) $G \trianglerighteq H_1 \trianglerighteq H_1 \cap K_1 = L \trianglerighteq L_1 ... \trianglerighteq L_t = 1$
(iii) $G \trianglerighteq K_1 \trianglerighteq H_1 \cap K_1 = L \trianglerighteq L_1 ... \trianglerighteq L_t = 1$
(iv) $G \trianglerighteq K_1 \trianglerighteq K_2 \trianglerighteq ... \trianglerighteq K_r = 1$

If we remove the first term G from (i) and (ii) we get two composition series for H_1. Since H_1 has order less than G, by our inductive hypothesis these two composition series are equivalent. Since the G we removed from the two series was the same, we conclude (i) and (ii) are two equivalent composition series for G. A similar argument shows the equivalence of (iii) and (iv). Composition series (ii) and (iii) are equivalent apart from their second term, they clearly have the same length and the same set of factors with only the first two interchanged so are equivalent composition series. It then follows (i) and (iv) are also equivalent. □

4.2. **An Alternative Proof of the Jordan-Holder Theorem.** We now state and prove Zassenhaus' Lemma and the Schreier Refinement Theorem. These two results can be used to give an alternative proof of the Jordan-Holder Theorem which does not require our group G to be finite.

First we state the definition of one normal series being a refinement as another as given in [Kna06, Chapter 9].

4.2.1. **Definition.** One normal series is said to be a **refinement** of another if the subgroups appearing in the second normal series all appear as subgroups in the first normal series.

4.2.2. **Lemma.** *(Zassenhaus' Lemma)*
Let H_1, H, K_1 and K be subgroups of G such that $H_1 \trianglelefteq H$ and $K_1 \trianglelefteq K$. Then,

(i) $H_1(H \cap K_1) \trianglelefteq H_1(H \cap K)$
(ii) $K_1(H_1 \cap K) \trianglelefteq K_1(H \cap K)$
(iii) $H_1(H \cap k)/H_1(H \cap K_1) \cong K_1(H \cap K)/K_1(H_1 \cap K)$

Our proof follows [Ros09, Lemma 9.7]

Proof. (i) Since $K_1 \trianglelefteq K$, $H \cap K_1 \trianglelefteq H \cap K$ using 2.1.6. Also, from 2.1, since $H_1 \trianglelefteq H$, $H_1(H \cap K_1) \trianglelefteq H_1(H \cap K)$ which is our required result.

(ii) Similar to (i), but with the roles of H_1 and H interchanged with K_1 and K respectively.

(iii) First we shall note some equations needed for our proof.

- From the Second Isomorphism Theorem 2.2.5, $H \cap K \leq H_1(H \cap K)$.
- By part (i) of this lemma we have $H_1(H \cap K_1) \trianglelefteq H_1(H \cap K)$.
- Also, since $H \cap K_1 \leq H \cap K$ as stated in part (i), $H_1(H \cap K) = H_1(H \cap K_1)(H \cap K)$.
- Also, $(H \cap K) \cap H_1(H \cap K_1) = (H_1 \cap K)(H \cap K)$.

We use these equations and apply the Second Isomorphism Theorem 2.2.5. Letting $G = H_1(H \cap K)$, $H = (H \cap K)$ and $N = H_1(H \cap K_1)$ in 2.2.5, we get

$$\begin{aligned} H_1(H \cap K)/H_1(H \cap K_1) &= H_1(H \cap K_1)(H \cap K)/H_1(H \cap K_1) \\ &\simeq H \cap K/(H \cap K) \cap (H_1(H \cap K_1)) \\ &= H \cap K/(H \cap K_1)(H_1 \cap K) \end{aligned}$$

Using (ii) we can make a similar argument interchanging H_1 and H with K_1 and K respectively throughout. This gives two identical equations and our result follows. □

4.2.3. **Theorem.** *(Schreier Refinement Theorem)*
Let $G = H_1 \trianglerighteq H_2 \trianglerighteq \ldots \trianglerighteq H_r = \{1\}$ and $G = K_1 \trianglerighteq K_2 \trianglerighteq \ldots \trianglerighteq K_s = \{1\}$ be two normal series of G. These series have equivalent refinements.

Proof. Following the proof of [Ros09, Theorem 9.8], we begin by constructing a new normal series $S_{1,2}$ by essentially inserting a copy of S_2 between each term in S_1. Similarly, we construct a second normal series, $S_{2,1}$ by inserting a copy of S_1 between each term in S_2.

For $0 \leq n \leq r$ and $0 \leq m \leq s$ let $I_{n,m}$ let us denote the subgroup $I_{n,m} = H_n(H_{n+1} \cap K_m)$. Then we note the following:

(1) $I_{n,0} = H_n(H_{n+1} \cap K_0) = H_n(H_{n+1} \cap \{1\}) = H_n$
(2) $I_{n,s} = H_n(H_{n+1} \cap G) = H_n H_{n+1} = H_{n+1}$ *(since* $H_n \leq H_{n+1}$*).*

(3) For $0 < m < s$, $I_{n,m} = H_n(H_{n+1} \cap K_m) \trianglelefteq H_n(H_{n+1} \cap K_{m+1}) = K_{n,m+1}$ by 4.2.2(i) where we have used $H_n = H_1$, $H_{n+1} = H$, $K_m = K_1$ and $K_{m+1} = K$.

Now we can write $S_{1,2}$ as $G = I_{r,s} \trianglerighteq ... \trianglerighteq I_{2,0} \trianglerighteq I_{1,s} \trianglerighteq ... \trianglerighteq I_{0,s} \trianglerighteq ... \trianglerighteq I_{0,0} = \{1\}$.

Similarly, for $S_{2,1}$ we let $J_{m,n} = K_m(K_{m+1} \cap H_n)$, which gives the refinement $G = J_{s,r} \trianglerighteq ... \trianglerighteq J_{1,0} \trianglerighteq J_{0,s} \trianglerighteq ... \trianglerighteq J_{0,0} = \{1\}$.

$S_{1,2}$ and $S_{2,1}$ have the same length of $rs + r + s$.

Now by 4.2.2 (iii), we get $I_{n,m+1}/I_{n,m} = H_n(H_{n+1} \cap K_{m+1})/H_n(H_{n+1} \cap K_m) \simeq K_m(K_{m+1} \cap H_{m+1})/K_m(K_{m+1} \cap H_n) = J_{m+1,n}/J_{m,n}$

Therefore the sets of factors of $S_{1,2}$ and $S_{2,1}$ are identical and we have shown that the refinements are equivalent. □

Proof. (*Alternative proof of the Jordan-Holder Theorem.*)
Suppose we have two composition series for G. These are normal series so by the Schreier Refinement Theorem they have equivalent refinements. Since they are composition series they have maximum length and so the only way to insert a new term into the series is to repeat a term . This results in a factor H_{j-1}/H_j isomorphic to $\{1\}$. So the number of terms added to each series is the same, as the factors from two refinements must match. Therefore our original two composition series must of had the same length and so we have shown their equivalence. □

References

[KM77] M. I. Kargapolov and Yu. I. Merzlyakov. *Osnovy teorii grupp*. Second. Izdat. "Nauka", Moscow, 1977, p. 240 (cit. on pp. 5, 7).

[Kna06] Anthony W. Knapp. *Basic algebra*. Cornerstones. Along with a companion volume ıt Advanced algebra. Birkhäuser Boston, Inc., Boston, MA, 2006, pp. xxiv+717. ISBN: 978-0-8176-3248-9; 0-8176-3248-4. DOI: 10.1007/978-0-8176-4529-8. URL: https://doi.org/10.1007/978-0-8176-4529-8 (cit. on pp. 3, 5, 8, 10).

[Mac12] Antonio Machì. *Groups*. Vol. 58. Unitext. An introduction to ideas and methods of the theory of groups. Springer, Milan, 2012, pp. xiv+371. ISBN: 978-88-470-2420-5; 978-88-470-2421-2. DOI: 10.1007/978-88-470-2421-2. URL: https://doi-org.manchester.idm.oclc.org/10.1007/978-88-470-2421-2 (cit. on pp. 2, 5, 6, 9).

[Ros09] H. E. Rose. *A course on finite groups*. Universitext. Springer-Verlag London, Ltd., London, 2009, pp. xii+311. ISBN: 978-1-84882-888-9. DOI: 10.1007/978-1-84882-889-6. URL: https://doi-org.manchester.idm.oclc.org/10.1007/978-1-84882-889-6 (cit. on pp. 2, 4–6, 9, 11).

[SS15] Ernest Shult and David Surowski. *Algebra—a teaching and source book*. Springer, Cham, 2015, pp. xxii+539. ISBN: 978-3-319-19733-3; 978-3-319-19734-0 (cit. on p. 7).