RAMSEY THEORY ON GRAPHS

MATH61000, MINI-PROJECT

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Student ID: XXXXXXXX

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Department of Mathematics, The University of Manchester

Contents

1. Introduction

Ramsey Theory, named after F. P. Ramsey, is the combinatorical study of finding certain substructures in large enough structures. In Ramsey's original paper, 'On A Problem Of Formal Logic', [8], he presents his original theorem on sets. However, Ramsey Theory extends to many areas of mathematics, such as geometry, number theory and logic.

In this text we will look at Ramsey theory on graphs. The reader will need a basic understanding in graphs and combinatorics. We will recap some definitions but for basic definitions, refer to Chapter 1 of [5].

An overview of the content of this project is as follows. In section 2 we will go through some preliminaries and introduce the notion of colouring the edges of graphs and what the Ramsey number is. A motivating example will be given to illustrate these notions. In section 3 we present our main theorem, 3.1, and give a proof of it, therefore showing existence of the Ramsey Number. Furthermore, we look at how the upper bound given to us via this proof corresponds to the actual value of certain Ramsey numbers. In section 4 we will look at the techniques used to find tighter bounds and lower bounds on the Ramsey number. We acknowledge the fact only few values of Ramsey numbers are known, and give some insight on why this may be.

The aim of this project is to highlight on the enigma of Ramsey Theory and how what may seem a simple problem is actually a topic that has perplexed mathematicians for many years.

2. Graph Colourings

In our first section we will talk about what it means to colour a graph. One may colour the edges or the vertices of a graph but in this project we are only going to talk about colouring the edges of a graph.

For notation purposes, let us first recall the definition of a graph, as stated in [5], page 2.

2.1. **Definition.** A *graph* is an ordered pair $G = (V, E)$ such that V is a set of vertices and $E \subseteq \{\{x, y\} \mid x, y \in V, x \neq y\}$ is the set of edges, where we may write xy for edge $\{x, y\}$. We refer to the vertex set of G as $V(G)$ and the edge set of G as $E(G)$.

We will assume that that the reader knows the terms adjacent, order of a graph and degree of a vertex. If the reader would like more information on basic definitions, refer to chapter 1 of [5].

2.2. **Definition.** A *complete graph* K_n is a graph such that $|V(G)| = n$ and $E(G) = \{\{x, y\} \mid x, y \in V, x \neq y\}$ and $|E(G)| = \binom{n}{2}$, so that the graph has n vertices and every vertex is joined to every other vertex in the graph with an edge, the maximum number of edges.

Now we introduce the notion of r-colouring, as stated in [5], page 96.

2.3. **Definition.** An *r-colouring* of of the edges of a graph $G = (V, E)$ is a function $c : E \to \{1, 2, ..., r\}$. In other words we are assigning a "colour" to each of the edges and the set of colours is $\{1, 2, ..., r\}$.

In this section and for the majority of this project we will be talking purely about two-colourings and taking the colours as red and blue. When we say that a graph or subgraph is monochromatic we mean that each of the edges of this graph are of the same colour.

Now we introduce the Ramsey Number for two-colourings on complete graphs, as stated in [4], pages 1-2.

2.4. **Definition.** The *Ramsey Number* $R(k, l)$, with $k, l \in \mathbb{Z}_+$ is the least positive integer, $n \in \mathbb{Z}_+$ such that any two-colouring of the edges of K_n contains either a monochromatic copy of K_k or a monochromatic copy of K_l.

Here we can see that $k, l \leq n$. Note that we are wanting a graph large enough so that *any* and all two-colourings of this graph will have a monochromatic subgraph on k or l vertices.

We may also look at the case where we are simply looking for a monochromatic copy of K_k in a large enough complete graph. In this case $R(k, k) = R(k)$ and this is the least positive integer $n \in \mathbb{Z}_+$ such that any two-colouring of the edges of this graph contains a monochromatic copy of K_k. Sometimes $R(k, k)$ is called the diagonal Ramsey Number.

The question is whether this number actually exists for $k, l \geq 2$. We will see in section 3 that it does in fact and we will see a proof of this.

We will now refer to [3], Chapter 13. One may want to look at the problem of finding monochromatic structures within larger structures within a real life setting. Consider a group of six people. Will there always be a group of three people that are either mutual acquaintances or mutual strangers? The answer is yes, and showing this is equivalent to proving that $R(3,3) \leq 6$. In the following example we will prove exactly this. Think about the people as vertices and we will join each person with a connection that is either acquaintance (red edge) or stranger (blue edge). What we have done is two coloured the graph K_6. Let us put the problem into a formal setting and prove it.

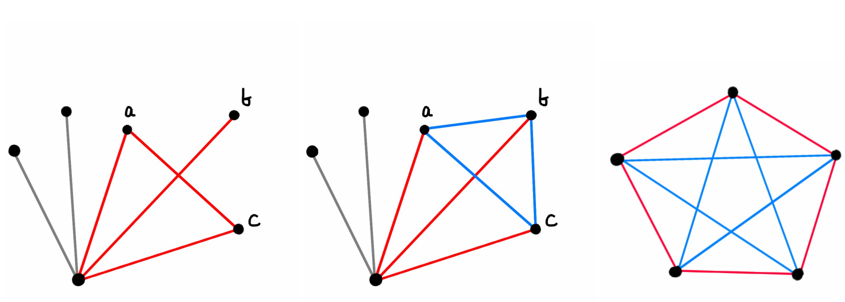
2.5. *Example.* Consider the complete graph on six vertices, K_6. Prove that upon two-colouring the edges of the graph, we may always find a monochromatic triangle, K_3.

Proof. Take a vertex $v \in V(K_6)$. The degree of this vertex is 5, as it is connected to all other 5 vertices. So these five edges will either have at least three edges of the same colour. Without loss of generality, assume there are three red edges which connect vertex v to vertices a, b and c. Look at figure 1a. We can see that if we choose to connect any of a, b or c to each other with a red edge then we immediately have a red triangle. So all these edges between these three vertices must be blue. But then by looking at figure 1b, we have made a blue triangle.

Hence, we will always find a monochromatic triangle in a two-coloured complete graph on 6 vertices and therefore $R(3,3) \leq 6$.

□

In fact, it is true that $R(3,3) = 6$, which we can see by simply showing that there exists no monochromatic triangle in a particular complete graph on 5 vertices, showing that $R(3,3) \geq 5$, see figure 1c.



(A) (B) (C)

FIGURE 1

This example should be regarded as motivation for what is to come. Although it may seem simple to work out these Ramsey Numbers, this is far from the case. The largest known value for $R(k, k)$, when $k = l$, is $R(4, 4) = 18$ (for further information look at Example 3.3). However, there are many bounds on these numbers, for example the value of $R(5, 5)$ is known to be between 43 and 48. We will look further into bounds on Ramsey Numbers in Section 4. The largest known value of $R(k, l)$ for $k \neq l$ is $R(3, 9) = 36$, which we note is still a small number. A table of known values and bounds for Ramsey number can be found in [7], page 4.

3. The Existence of the Ramsey Number

So as we have seen in section 2, we have some values for Ramsey Numbers and certain methods to find them. In this section we show that in fact the Ramsey Number exists for all $k, l \geq 2$, even if we don't know what the value is. Now we present and prove the main theorem of the project, using [3] as reference.

3.1. **Theorem.** The Ramsey Number, $R(k, l)$, defined in 2.4 exists for all $k, l \geq 2$ and is such that

$$R(k, l) \leq R(k, l - 1) + R(k - 1, l). \tag{3.1}$$

Proof. This is a proof by induction where we show existence of $R(k, l)$. For the base case look at $R(k, 2)$. So this means we are either looking for a red K_k or a blue K_2. We have $R(k, 2) \geq k$, as there is no red K_k or blue K_2 in K_{k-1} where all the edges are red. We also have $R(k, 2) \leq k$ as if we have a two-colouring such that all the edges are red then we have a red K_k, and in all other cases of colourings there is at least one blue edge, which is exactly a blue K_2. So $R(k, 2) = k$. Using the same argument, we have that $R(2, l) = l$. So $R(k, l)$ exists for $l = 2$ and $k = 2$.

For the induction step we assume that $R(k - 1, l)$ and $R(k, l - 1)$ exist for $k, l \geq 2$ and use this to prove that $R(k, l)$ exists for all $k, l \geq 2$ and indeed that $R(k, l) \leq R(k, l - 1) + R(k - 1, l)$.

Let us take a complete graph, G, with $R(k, l-1)+R(k-1, l)$ vertices. Note that any vertex v of a complete graph, K_n, has degree $n - 1$.

Now, let us take an arbitrary vertex $v \in V$. Let R_v and B_v be such that

$$R_v = \{y \in \{v, y\} \mid \{v, y\} \in E(G), \{v, y\} \text{ is red}\},$$
$$B_v = \{y \in \{v, y\} \mid \{v, y\} \in E(G),\ \{v, y\} \text{ is blue}\}.$$

These are sets of adjacent vertices to v, such that R_v is the set of vertices, y, such that v is joined to y with a red edge, and R_v is the set of vertices, y, such that v is joined to y with a blue edge. So as the degree of v is $R(k, l-1) + R(k-1, l) - 1 = |R_v| + |B_v|$, we have that either $|R_v| \geq R(k-1, l)$ or $|B_v| \geq R(k, l-1)$.

If we look at the first case where $|R_v| \geq R(k-1, l)$. R_v is the vertex set of endpoints of the red edges of v. This forms a complete graph with order at least $R(k-1, l)$. So by the inductive hypothesis, this complete graph contains either a blue K_l, in which case we are done, or a red K_{k-1}. If we have a red K_{k-1}, then add vertex v and all edges joining v to the vertices of this K_{k-1}. Adding this vertex makes a complete graph on k edges, and as all the edges we are adding are red, we have made a red K_k.

Similarly if $|B_v| \geq R(k, l-1)$, we will either have a red K_k, in which case we are done or we will have a blue K_{l-1} which we may extend to a blue K_l with the addition of vertex v and all blue edges joining v to this blue K_{l-1}.

So in either case we either have a red K_k or a blue K_l, and so by definition of the Ramsey number 2.4, $R(k, l) \leq R(k, l-1) + R(k-1, l)$ and so it exists.

□

This proof not only proves that the Ramsey Number exists for all $k, l \geq 2$, but also gives us an upper bound. In section 4, we will look further into bounds on Ramsey Numbers and how upper and lower bounds can be constructed.

For now, we will look at how our current upper bound from theorem 3.1 can aid us in finding values of $R(k, l)$.

3.2. *Example.* In the proof of theorem 3.1, we saw that $R(k, 2) = R(2, k) = k$ for all $k \geq 2$. So we have that $R(4, 2) = 4$. Recall from example 2.5 that $R(3, 3) = 6$. So, with these values we may find an upper bound for $R(4, 3)$. Inserting our values of $k = 4, l = 3$ into equation 3.1, we get

$$R(4, 3) \leq R(4, 2) + R(3, 3) = 4 + 6 = 10.$$

This upper bound is close because in fact we have $R(4, 3) = 9$. The proof of $R(4, 3) \leq 9$ is given by an analysis of the graph K_9, showing in a similar way to Example 2.5 that there must always exist a red K_4 or a blue K_3. For further information see Example 13.3 of [3].

In Section 4, Example 4.3, we provide a proof of $R(4, 3) \geq 9$.

Using the value of $R(4, 3) = 9$, it is now easy to show the following.

3.3. *Example.* We show that $R(4,4) \leq 18$, given the information that $R(4,3) = R(3,4) = 9$ from Example 3.2. Inserting our values of k and l into equation 3.1 we get

$$R(4,4) \leq R(4,3) + R(3,4) = 18$$

So we have that $R(4,4) \leq 18$. It can be shown that $R(4,4) \geq 18$ by analysing a specifically two-coloured K_{17}, for more information read Example 13.4 of [3].

4. Bounds on Ramsey Numbers

As we have commented on, there is no formula to work out values of Ramsey Numbers. However, there are bounds that can allow us close to the actual value.

For further reading on recent developments on upper and lower bounds on the Ramsey number, the survey by Conlon, Fox and Sudakov [4] is a very interesting read, but these will largely not be looked at in this text. For now we will look at consequences of the upper bound developed in Theorem 3.1, that are slightly tighter upper bounds. Furthermore, we will look at one of the first lower bounds and give an explanation of the proof.

The following theorem and proof will follow how it is presented in [3], original source [6].

4.1. **Theorem.** Let $k, l \geq 2$ with $k, l \in \mathbb{Z}$. Then

$$R(k,l) \leq \binom{k+l-2}{k-1} \tag{4.1}$$

Proof. This, again, is a proof by induction on on k and l.

For the base case, if $k = 2$, then we have that $R(2,l) \leq \binom{l}{1} = l$. Recall from the proof of 3.1 we showed that $R(2,l) = l$, so the base case holds for $k = 2$ with equality in fact. If $l = 2$, we similarly have $R(2,l) \leq \binom{l}{1} = l$, which is again true and holds with equality by an analogous argument. So the base case holds.

Suppose the statement is true for $R(k, l-1)$ and $R(k-1, l)$ for all $k, l \geq$. By equation 3.1 we know that

$$R(k,l) \leq R(k,l-1) + R(k-1,l).$$

Applying the inductive hypothesis we get

$$\begin{aligned} R(k,l) &\leq R(k,l-1) + R(k-1,l) \\ &\leq \binom{k+(l-1)-2}{k-1} + \binom{(k-1)+l-2}{(k-1)-1} \\ &= \binom{k+l-3}{k-1} + \binom{k+l-3}{k-2} \\ &= \binom{k+l-2}{k-1}. \end{aligned}$$

In the last step we have used Pascal's Identity (see [3], Theorem 4.3, page 66):

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Hence, we have proved the result for all $k, l \geq 2$. □

From this upper bound we also get a very neat upper bound for the diagonal Ramsey Number $R(k,k)$.

4.2. **Corollary.** For all $k \geq 2$, $k \in \mathbb{Z}$ we have that $R(k,k) \leq 4^{k-1}$.

Proof. By substituting $k = l$ into equation 4.1, we have that

$$R(k,k) \leq \binom{2k-2}{k-1}$$

Now recall that the total number of subsets of a set of size n is 2^n. By the definition of the binomial coefficient we are choosing subsets of a set of size $2k-2$. Therefore,

$$R(k,k) \leq \binom{2k-2}{k-1} \leq 2^{2k-2} = 2^{2(k-1)} = 4^{k-1}.$$

□

For lower bounds the problem is different, we are instead trying to find two-coloured complete graphs where there does not exist a monochromatic K_k or K_l, some $k, l \geq 2$. The way to usually do this for specific values of k and l is to just find a graph on a certain number of vertices with no monochromatic K_k or K_l. The following example provides a proof of a lower bound for $R(4,3)$, using Example 13.4 of [3] as reference.

4.3. *Example.* Show that $R(4,3) \geq 9$.

Proof. To see that $R(4,3) \geq 9$, consider the following two-coloured K_8. Label the vertices of the graph clockwise with elements of $\{1,2,\ldots,8\}$ in ascending order. Let edge $ij \in E(K_8)$, with $i < j$ be blue if $i - j \in \{1,4,7\}$ and red if $i - j \in \{2,3,5,6\}$. Suppose there exists a blue triangle. Then take a smallest vertex, i, of this triangle. Vertex i must make a blue triangle with two of the vertices $i+1, i+4$ and $i+7$ for these edges to be blue. However, observe that none of these three vertices have difference $1, 4$ or 7, and so cannot have a blue edge between them, a contradiction. Hence there is no blue triangle.

On the other hand, suppose we have a red K_4. Take smallest vertex i of this red K_4. We would then need three of the four vertices $i+2, i+3, i+5$ and $i+6$. Observe that we must chose $i+2$ and $i+3$ together, or $i+5$ and $i+6$ together. However, their difference is 1 and so would be joined with a blue edge, a contradiction. Hence there is no red K_4. □

The brute force method of finding a complete graph such as in Example 4.3 works for when looking for bounds on a specific Ramsey number. However, there are more generalised results.

The following theorem we give with only an outline of the proof. The theorem gives a neat lower bound for the diagonal Ramsey Number and a proof may be found in [3], Theorem 15.1, page 343.

4.4. **Theorem.** For all $k \geq 3$, $k \in \mathbb{Z}$, we have that $R(k,k) > 2^{\frac{k}{2}}$.

The proof of this theorem is quite accessible and uses the probabilistic method, where it is proved that the probability of finding no monochromatic K_k in a larger two-coloured K_n cannot be zero, implying there is at least one outcome with no monochromatic K_k. This is one of the simpler proofs of a lower bound.

Although it is a general lower bound, it is not necessarily a very good one. To give a feel of this lower bound let us look at how it compares to known values.

4.5. *Example.* The lower bound in Theorem 4.4 gives $R(4,4) > 2^{\frac{4}{2}} = 4$. As we know that $R(4,4) = 18$, this lower bound is not as revealing as hoped.

For other upper and lower bounds a lot more work is required. The best known bounds for $k \neq l$ are provided in the following theorem, referring to [4].

4.6. **Theorem.** For any $k, l \geq 2$ there exist positive constants c_k, c'_k such that

$$c'_k \frac{l^{\frac{k+1}{2}}}{(\log l)^{\frac{k+1}{2} - \frac{1}{k-2}}} \leq r(k, l) \leq c_k \frac{l^{k-1}}{(\log l)^{k-2}}.$$

The lower bound here was proved by Bohman and Keevash in 2010 [2], and upper bound by Ajtai, Komlós and Szemeréedi in 1980 [1]. As we can see these are not the easiest of bounds to understand and the proofs of them require a lot of work. Mathematicians are still no closer to finding a formula for the Ramsey Number and, using [4] as a reference, it shows improvements to the bounds are not being made at rapid pace.

A formula for $R(k, l)$ is an active area of research to this day and proves to be a rather elusive problem.

5. Conclusion

In this project we have studied Ramsey Theory on graphs, where the beauty of Ramsey Theory is possibly best illustrated. We have seen examples of many methods of finding Ramsey Numbers, a complete proof of the existence of the Ramsey Number, $R(k, l)$, for any $k, l \geq 2$ and also looked at how bounds on the Ramsey Number are made.

Although we have only looked at two-colouring complete graphs, Theorem 3.1 can be generalised for r-colourings for any $r \geq 2$. In fact, Ramsey Theory can be applied to many areas of mathematics, one of the most interesting being convex geometry. Consider finding a set of points in the plane large enough so that n of these points lie in convex position. This is what is referred to as the Happy Ending Problem, and follows on very nicely from what has been studied in this project. Erdös and Szekeres's paper, [6], on this very problem is also where the bound in Theorem 4.1 was first presented. If the reader would like to further explore the applications of Ramsey Theory, Vera Rosta's survey [9], takes a comprehensive view of this subject.

This project has given the reader a taste of Ramsey Theory, and insight into areas of this combinatorical problem that remain an unsolved puzzle.

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